CEGEP JOHN ABBOTT COLLEGE

LINEAR ALGEBRA
(MATHEMATICS 201-NYC-05/∞)

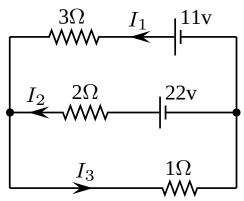
1. Solve, if possible.

   (a) $\begin{aligned} 3x_1 - 2x_2 + x_3 - x_4 &= 5 \\ 6x_1 - 4x_2 - x_3 + 2x_4 &= 7 \\ 9x_1 - 6x_2 \qquad + x_4 &= 12 \end{aligned}$

   (b) $\begin{aligned} 2x_1 + 2x_2 - 4x_3 + 8x_4 &= 6 \\ x_1 - 2x_2 + 7x_3 - 2x_4 &= 9 \\ 3x_1 + x_2 \qquad + 8x_4 &= 5 \end{aligned}$

2. For what value(s) of k, if any, does the system $\begin{cases} 2x - 3y = 6 \\ kx + 2y = -4 \end{cases}$ have (a) exactly one solution? (b) no solution? (c) infinitely many solutions?

3. Find the currents in the following circuit.



4. Evaluate $AB - I$ where $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 \\ 6 & -1 \\ 0 & 2 \end{pmatrix}$.

5. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find a solution other than $B = I$ or $B = 0$ to the matrix equation $AB = BA$.

6. Let $A = \begin{pmatrix} 1 & k \\ -1 & 1 \end{pmatrix}$
   (a) Evaluate A^TA.
   (b) For what values of k does $(A^TA)^{-1}$ exist?

7. Assume A is a square matrix such that $A^2 = 0$. Prove that $(I + A)^{-1} = I - A$.

8. Let $A = \begin{pmatrix} 0 & 2 & -3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$
   (a) Find A^{-1}
   (b) If $B = \frac{1}{2}A$ find B^{-1} using this equation.

9. Assume that $A = LU = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Use this LU factorization of A without finding A to solve $A\vec{x} = (\,1 \;\; -3 \;\; 0\,)^T$.

10. Assume that A is a 4×4 matrix such that $\det(A) = -2$. Find
    (a) $\det\left(-\frac{1}{2}A\right)$, (b) $\det\left(A^{-1}\right)$, (c) $\det(AA^T)$,
    (d) $\det(R)$, where R is the reduced row echelon form of A.

11. Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 4 & 0 \\ 1 & 2 & 0 \end{pmatrix}$
    (a) Write A^{-1} as a product of elementary matrices.
    (b) Write A as a product of elementary matrices.

12. Evaluate $\det \begin{pmatrix} 3 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$.

13. If $\vec{u}$ and $\vec{v}$ are perpendicular, prove that $(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2$.

14. Given points $A(1, 0, -2)$, $B(0, 2, 5)$, $C(4, 2, -7)$ and $D(3, -2, 4)$, find
    (a) the area of $\triangle ABC$,
    (b) the angle A of $\triangle ABC$, in degrees, correct to 2 decimal places,
    (c) the volume of the parallelepiped formed by the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$,
    (d) the point P such that $\overrightarrow{AP} = \overrightarrow{BC}$,
    (e) a line perpendicular to the plane $x - 2y + 3x = 6$ and containing the point A,
    (f) the distance from the point B to the plane $x - 2y + 3z = 6$.

15. Given the lines ℓ_1 and ℓ_2,
    $$\ell_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \quad \ell_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 7 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$
    find
    (a) the point of intersection of ℓ_1 and ℓ_2,
    (b) the equation of the plane containing ℓ_1 and ℓ_2,
    (c) the intersection (if any) of the line ℓ_1 and the plane $3x - 4y + 7z = 48$.

16. Let $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : y = x + 2z \right\}$. Is S (a) closed under scaling? (b) closed under addition? (c) a subspace of $\mathbb{R}^3$? Justify all assertions.

17. Determine if the set $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix} \right\}$ is linearly independent or linearly dependent.

18. Given the set $S = \left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.
    (a) Prove that S is a linearly independent set.
    (b) Explain why S is not a basis for $\mathbb{R}^3$.
    (c) Give a basis for $\mathbb{R}^3$ which includes the two vectors in the set S, and justify your claim that you now have a basis for $\mathbb{R}^3$.

19. Give a geometric and algebraic description of the subspace spanned by
    (a) $S_1 = \left\{ \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \right\}$ (b) $S_2 = \left\{ \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \right\}$
    (c) $S_3 = \left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ -6 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \\ 9 \end{pmatrix} \right\}$

20. Let $\{\vec{v}_1, \vec{v}_2\}$ be a basis for $\mathbb{R}^2$. Let $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$ be any vector in $\mathbb{R}^2$. Show that this representation of $\vec{x}$ in terms of v_1 and $\vec{v}_2$ is unique.

21. Given $A = \begin{pmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & 4 \\ 3 & 0 & -9 & 1 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & 9 & 0 & -4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ determine a basis for, and the dimension of
    (a) the nullspace of A,
    (b) the row space of A,
    (c) the column space of A, and
    (d) give the rank of A.