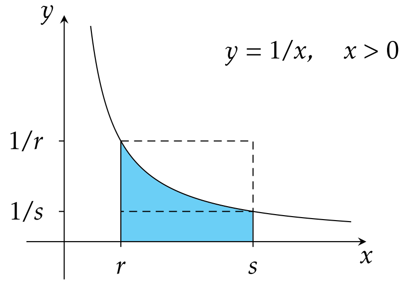
Logarithmic and exponential functions

Calculus I

201-NYA-05

Area under the hyperbola

For positive real numbers r and s with $r < s$, let $A(r,s)$ denote the area of the region $\{(x,y)\colon r \leqslant x \leqslant s,\ 0 \leqslant y \leqslant 1/x\}$, as shaded in the figure below.



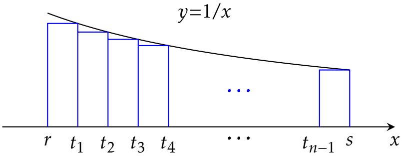
The shaded region includes a rectangle of height $1/s$ and base $s-r$, and is included in a rectangle of height $1/r$ and base $s-r$. Therefore,

$$\frac{s-r}{s} < A(r,s) < \frac{s-r}{r}. \tag{1}$$

Area under the hyperbola

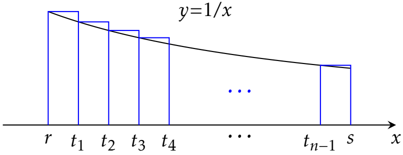
If $[r, s]$ is divided into n parts of equal length by numbers $t_1, t_2, \ldots, t_{n-1}$, where $r < t_1 < t_2 < \cdots < t_{n-1} < s$, then the length of each part is $(s-r)/n$.

The heights of the rectangles in the lower estimates are $\dfrac{1}{t_1}, \dfrac{1}{t_2}, \dfrac{1}{t_3}, \ldots, \dfrac{1}{s}$, so a lower estimate of the total area is given by



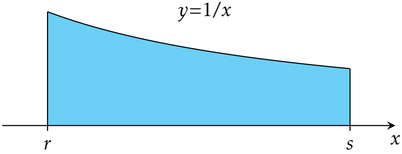
$$L_n = \frac{s-r}{n}\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{s}\right).$$

The heights of the rectangles in the upper estimates are $\dfrac{1}{r}, \dfrac{1}{t_1}, \dfrac{1}{t_2}, \ldots, \dfrac{1}{t_{n-1}}$, so an upper estimate of the total area is given by



$$U_n = \frac{s-r}{n}\left(\frac{1}{r} + \frac{1}{t_1} + \cdots + \frac{1}{t_{n-1}}\right).$$

Area under the hyperbola



$A(r,s)$ = the area of the region

$$\{(x,y)\colon\ r \leqslant x \leqslant s,\ 0 \leqslant y \leqslant 1/x\}.$$

From the previous page there are estimates

$$L_n < A(r,s) < U_n$$

of the area $A(r,s)$, where

$$L_n = \frac{s-r}{n}\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{s}\right) \quad \text{and} \quad U_n = \frac{s-r}{n}\left(\frac{1}{r} + \frac{1}{t_1} + \cdots + \frac{1}{t_{n-1}}\right).$$

Since

$$0 < U_n - L_n = \frac{s-r}{n}\left(\frac{1}{r} - \frac{1}{s}\right) = \frac{(s-r)^2}{rs} \cdot \frac{1}{n}, \qquad (2)$$

the estimates give concrete meaning to $A(r,s)$, and a means of approximating its value to any desired accuracy. Even if the parts have different lengths, the difference between the upper and lower estimates of $A(r,s)$ remains bounded by the right side of (2) provided the length of each part is $\leqslant (s-r)/n$.

Area under the hyperbola (example)

As an illustration let us approximate $A(1,2)$.

The interval $[\,1,2\,]$ is divided into 5000 parts of equal length by the numbers

$$1 < \frac{5001}{5000} < \frac{5002}{5000} < \frac{5003}{5000} < \cdots < \frac{9999}{5000} < 2.$$

The corresponding lower estimate is

$$L = \frac{1}{5000}\left(\frac{5000}{5001} + \frac{5000}{5002} + \cdots + \frac{1}{2}\right) = \frac{1}{5001} + \frac{1}{5002} + \cdots + \frac{1}{10{,}000} \approx 0.693097,$$

and the corresponding upper estimate is

$$U = \frac{1}{5000}\left(1 + \frac{5000}{5001} + \cdots + \frac{5000}{9999}\right) = \frac{1}{5000} + \frac{1}{5001} + \cdots + \frac{1}{9999} \approx 0.693197.$$

So $0.693097 < A(1,2) < 0.693197$, which gives three decimal places of $A(1,2)$.

Area under the hyperbola (example, continued)

Certain collections of approximations may be interpreted as limits.

The interval $[1,2]$ is divided into n parts of equal length by the numbers

$$1 < \frac{n+1}{n} < \frac{n+2}{n} < \cdots < \frac{2n-1}{n} < 2.$$

The lower estimate is

$$L_n = \frac{1}{n}\left(\frac{n}{n+1} + \frac{n}{n+2} + \cdots + \frac{1}{2}\right) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n},$$

and the upper estimate is

$$U_n = \frac{1}{n}\left(1 + \frac{n}{n+1} + \cdots + \frac{n}{2n-1}\right) = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}.$$

Since

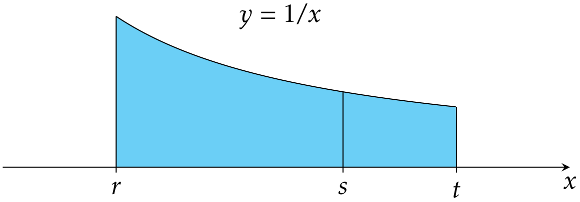
$$L_n < A(1,2) < U_n \qquad \text{and} \qquad U_n - L_n = \frac{1}{2n},$$

it follows that $\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = A(1,2)$. The former limit, written explicitly, is

$$\lim_{n\to\infty}\left\{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right\} = A(1,2).$$

Properties of the area $A(r,s)$

Let r, s and t be real numbers such that $0 < r < s < t$, and consider the areas $A(r,s)$, $A(s,t)$ and $A(r,t)$.



The figure suggests that

$$A(r,s) + A(s,t) = A(r,t). \tag{3}$$

This identity is an apparent property of area, and more concretely it is a direct consequence of the remarks about upper and lower estimates made on page 4.

Since $A(s,t) > (t-s)/t > 0$ and $A(r,s) > (s-r)/s > 0$, this last identity implies that

$$A(r,s) < A(r,t) \qquad \text{and} \qquad A(s,t) < A(r,t). \tag{4}$$

Properties of the area $A(r,s)$

Suppose that $0 < r < s$, and let the interval $[\,r, s\,]$ be divided into n parts of equal length by $r < t_1 < t_2 < \cdots < t_{n-1} < s$, with L_n, U_n the associated lower and upper estimates of $A(r,s)$.

Multiplying by any positive real number a yields $ar < at_1 < at_2 < \cdots < at_{n-1} < as$, which divide the interval $[\,ar, as\,]$ into n parts of length $(as - ar)/n = a(s-r)/n$. If L'_n, U'_n denote the associated lower and upper estimates of $A(ar, as)$, then

$$L'_n = \frac{a(s-r)}{n}\left(\frac{1}{at_1} + \frac{1}{at_2} + \cdots + \frac{1}{as}\right) = \frac{s-r}{n}\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{s}\right) = L_n$$

and

$$U'_n = \frac{a(s-r)}{n}\left(\frac{1}{ar} + \frac{1}{at_1} + \cdots + \frac{1}{at_n}\right) = \frac{s-r}{n}\left(\frac{1}{r} + \frac{1}{t_1} + \cdots + \frac{1}{t_{n-1}}\right) = U_n.$$

Therefore,

$$L_n < A(r,s) < U_n \qquad \text{and} \qquad L_n < A(as, ar) < U_n,$$

for $n = 1, 2, 3, \&c.$, so that (2) on page 4 implies that

$$A(ar, as) = A(r,s). \tag{5}$$

Properties of the area $A(r, s)$

Suppose that $x > 1$ and $y > 1$.

The identity (5) on page 8 implies (multiplying by x) that $A(1, y) = A(x, xy)$, and addition of areas (which is (3) on page 7) yields $A(1, x) + A(x, xy) = A(1, xy)$. Therefore,

$$A(1, xy) = A(1, x) + A(1, y). \tag{6}$$

If n is a positive integer, then repetition of this last identity (with $x^n = x \cdot x \cdots x$ and $x = x^{1/n} \cdot x^{1/n} \cdots x^{1/n}$) gives $A(1, x^n) = nA(1, x)$ and $A(1, x) = nA(1, x^{1/n})$. Therefore, if r is any positive rational number then

$$A(1, x^r) = rA(1, x). \tag{7}$$

Since (again multiplying by x and using identity (5) on page 8)

$$A(1/x, 1) = A(1, x), \tag{8}$$

the identities (6) and (7) have analogues in which $0 < x < 1$, or $0 < y < 1$, or r is a negative rational number.

Properties of the area $A(r,s)$

Since $A(1,2) > \frac{1}{2}$ and $nA(1,2) = A(1,2^n) = A(2^{-n},1)$, it follows that

$$A(2^{-n},1) > \tfrac{1}{2}n \qquad \text{and} \qquad A(1,2^n) > \tfrac{1}{2}n,$$

and therefore (by (4) on page 7)

$$\lim_{x\to 0^+} A(x,1) = \infty \qquad \text{and} \qquad \lim_{x\to\infty} A(1,x) = \infty. \tag{9}$$

The simplest estimate of $A(r,s)$ for $0 < r < s$ (*viz.* (1) on page 2) is equivalent to

$$\frac{1}{s} < \frac{A(r,s)}{s-r} < \frac{1}{r}.$$

From this it is immediate that

$$\lim_{r\to s^-} \frac{A(r,s)}{s-r} = \frac{1}{s} \qquad \text{and} \qquad \lim_{s\to r^+} \frac{A(r,s)}{s-r} = \frac{1}{r}. \tag{10}$$

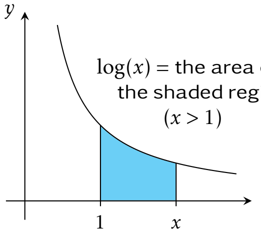
Definition of the logarithm

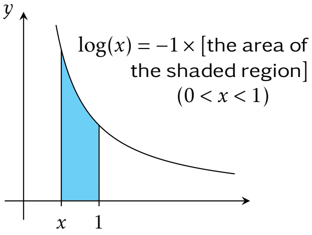
The **logarithm** of a positive real number x is denoted by $\log(x)$, or $\ln(x)$, and is defined by

$$\log(x) = \begin{cases} -A(x,1) & \text{if } 0 < x < 1, \\ 0 & \text{if } x = 1 \text{ and} \\ A(1,x) & \text{if } x > 1. \end{cases}$$

The logarithm is illustrated below, in the cases $x > 1$ and $0 < x < 1$.

$$y = 1/x, \quad x > 0$$





Sample computations — approximation and limits

The computation on page 5 implies that

$$0.693097 < \log(2) < 0.693197.$$

The logarithm of any positive real number may be approximated as on page 5, provided the sign of the result is observed in case $0 < x < 1$.

The last formula on page 6 is equivalent to

$$\log(2) = \lim_{n\to\infty}\left\{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right\}.$$

Exercise

By adapting the idea used to derive the last formula on page 6, show that if k and ℓ are integers, and p and q are positive integers such that $1 \leqslant q < p$, then

$$\log(p/q) = \lim_{n\to\infty}\left\{\frac{1}{qn+k} + \frac{1}{qn+k+1} + \frac{1}{qn+k+2} + \cdots + \frac{1}{pn+\ell}\right\}. \tag{11}$$

Properties of the logarithm (arithmetical)

The properties (6), (7) and (8) of $A(r,s)$ on page 9 give

$$\log(xy) = \log(x) + \log(y) \qquad \text{and} \qquad \log(x^r) = r\log(x), \tag{12}$$

where $x, y > 0$ and r is a rational number. The limits (9) on page 10 are

$$\lim_{x\to 0^+} \log(x) = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} \log(x) = \infty. \tag{13}$$

The basic estimate of $A(r,s)$ for $0 < r < s$, *viz.* (1) on page 2, is equivalent to

$$1 - \frac{1}{x} < \log(x) < x - 1 \qquad \text{if} \quad x > 0 \quad \text{and} \quad x \neq 1. \tag{14}$$

In particular, $0 < \log\left(x^{\beta/(2\alpha)}\right) < x^{\beta/(2\alpha)}$ for positive rational numbers α and β; applying (12) and rearranging then gives

$$0 < \frac{(\log x)^\alpha}{x^\beta} < \left(\frac{2\alpha}{\beta}\right)^\alpha \cdot x^{-\frac{1}{2}\beta}.$$

Thus, for any rational number α and any positive rational number β,

$$\lim_{x\to\infty} \frac{(\log x)^\alpha}{x^\beta} = 0 \qquad \text{and} \qquad \lim_{z\to 0^+} \left\{z^\beta(-\log z)^\alpha\right\} = 0. \tag{15}$$

The second limit is an equivalent form of the first, in which $z = 1/x$.

Properties of the logarithm (analytical)

The limits (10) on page 10 imply that

$$\frac{d}{dx}\Big\{\log(x)\Big\} = \lim_{x'\to x} \frac{\log(x') - \log(x)}{x' - x} = \frac{1}{x} \tag{16}$$

and in particular

$$\lim_{x\to 1} \frac{\log(x)}{x-1} = \lim_{t\to 0} \frac{\log(1+t)}{t} = 1. \tag{17}$$

So the logarithm is continuous (for it is differentiable) and, since $\log(x) \to \infty$ as $x \to \infty$ and $\log(x) \to -\infty$ as $x \to 0^+$, the range of the logarithm is $\mathbb{R}$.

Addition of areas, or more precisely property (4) on page 7, implies that

$$\log(x) < \log(y) \qquad \text{if, and only if,} \qquad 0 < x < y. \tag{18}$$

It follows that $\log(x) = 1$ has a unique solution, which is denoted by e. Since $1 < \left(1 + \frac{1}{n}\right)^n \leqslant 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} < 3$ for $n = 1, 2, 3, \&c.$, property (14) of the logarithm (on page 13) implies that the limit (17) is equivalent to

$$e = \lim_{n\to\pm\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{t\to 0} (1+t)^{1/t}, \tag{19}$$

in which $t = 1/n$ (in the second limit irrational exponents are anticipated).

The exponential function

Since $\log(x) < \log(y)$ if, and only if, $0 < x < y$, the logarithm has an inverse. The inverse of the logarithm, called the **exponential function**, is denoted by $\exp$ and defined by

$$y = \exp(x) \qquad \text{if, and only if,} \qquad \log(y) = x. \tag{20}$$

Equivalently,

$$\log(\exp x) = x \quad \text{for} \quad x \in \mathbb{R}, \qquad \text{and} \qquad \exp(\log x) = x \quad \text{for} \quad x > 0.$$

The domain of $\exp$ is $\mathbb{R}$, and the range of $\exp$ is $(0, \infty)$.

If $y = \exp(x)$ and $y' = \exp(x')$, then $x = \log(y)$ and $x' = \log(y')$, so the limit (16) on the previous page gives

$$\lim_{x' \to x} \frac{\exp(x') - \exp(x)}{x' - x} = \lim_{y' \to y} \frac{y' - y}{\log(y') - \log(y)} = y = \exp(x).$$

Therefore,

$$\frac{d}{dx}\Big\{\exp(x)\Big\} = \exp(x), \tag{21}$$

and in particular

$$\lim_{t \to 0} \frac{\exp(t) - 1}{t} = 1. \tag{22}$$

Properties of the exponential function (arithmetical)

Via the definition (20) on the preceding page, the properties of the logarithm on page 13 translate directly into properties of the exponential function.

For real numbers x, y, and any rational number r, the identities (12) become

$$\exp(x+y) = \exp(x)\exp(y) \qquad \text{and} \qquad \exp(rx) = (\exp x)^r, \tag{23}$$

and the latter defines its right side if r is irrational. The limits (13) become

$$\lim_{x\to-\infty} \exp(x) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} \exp(x) = \infty. \tag{24}$$

The inequality (14) implies that

$$1 + x < \exp(x) \qquad \text{if} \quad x \neq 0. \tag{25}$$

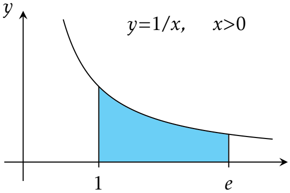
The limit (15) becomes (with $y = (\log x)^{1/c}$, or $x = e^{y^c}$, so that $a = \alpha c$ and $b = \beta$)

$$\lim_{y\to\infty} \frac{y^a}{\exp(by^c)} = 0, \tag{26}$$

where b and c are positive rational numbers, and a is any rational number.

The number e

Recall from page 14 that e is the unique real number whose logarithm is 1.



The area of the shaded region is 1; *i.e.*, $\log(e) = 1$. From (19) on page 14,

$$e = \lim_{n\to\pm\infty}\left(1+\frac{1}{n}\right)^n = \lim_{t\to 0}(1+t)^{1/t}.$$

Expanding the power in the first limit and revising the terms gives

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1 + n\cdot\frac{1}{n} + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{1}{n^2} + \cdots + \frac{n}{1}\cdot\frac{n-1}{2}\cdots\frac{n-\nu+1}{\nu}\cdot\frac{1}{n^\nu} + \cdots + \frac{1}{n^n}\\ &= 1 + 1 + \frac{1}{2}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{\nu!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{\nu-1}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right).\end{aligned}$$

It follows that $\left(1+\frac{1}{n}\right)^n < 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n!} < \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, and therefore

$$e = \lim_{n\to\infty}\left\{1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots + \frac{1}{n!}\right\}. \tag{27}$$

This limit is well suited for approximating e. For example, taking a sum of twenty-five terms ($n = 24$) gives $e \approx 2.7182818284590452353602874$, which is correct up to the twenty-fifth place after the decimal point.

The exponential function — additional formulæ

The notation e^x for $\exp(x)$ extends the usual notation for rational exponents, by (23) on page 16. In this notation the functional definition is $y = e^x$ if, and only if, $\log(y) = x$. Equivalently, $\log(e^x) = x$ for $x \in \mathbb{R}$, and $e^{\log(x)} = x$ for $x > 0$.

The arithmetical properties of the exponential function are then

$$e^{x+y} = e^x e^y, \quad e^{xy} = (e^x)^y, \quad \lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to-\infty} e^x = 0, \quad \lim_{x\to\infty} \frac{x^a}{e^{bx^c}} = 0$$

for a real and $b, c > 0$, and $e^x > 1 + x$ for $x \neq 0$. The derivative and basic limit are

$$\frac{d}{dx}(e^x) = e^x \qquad \text{and} \qquad \lim_{t\to 0} \frac{e^t - 1}{t} = 1.$$

If $t' = xt$ then $(1 + xt)^{1/t} = \left((1 + t')^{1/t'}\right)^x$, so the limit (19) on page 14 yields

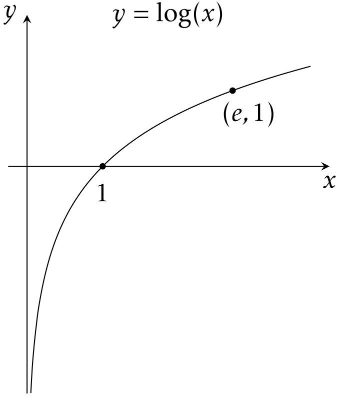
$$e^x = \lim_{t\to 0}(1 + xt)^{1/t} = \lim_{n\to\pm\infty}\left(1 + \frac{x}{n}\right)^n. \tag{28}$$

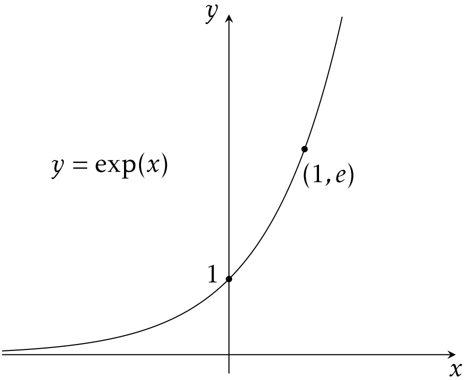
The computations on page 17, with $\left(1 + \frac{x}{n}\right)^n$ in place of $\left(1 + \frac{1}{n}\right)^n$, then yield

$$e^x = \lim_{n\to\infty}\left\{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots + \frac{x^n}{n!}\right\}. \tag{29}$$

Graphs of the logarithm and the exponential function

Below are graphs of the logarithm and the exponential function, with two points of interest on each graph labelled and emphasized. Although nothing mathematical can be deduced from a graph, the images may facilitate the recall of certain basic features of each function.





Logarithmic/exponential functions with other bases

For $a > 0$ and $a \neq 1$, the *logarithmic and exponential functions with base a* are

$$\log_a(x) = \frac{\log(x)}{\log(a)} \qquad \text{and} \qquad a^x = \exp(x \log a).$$

By the identities (12) on page 13, the arithmetical properties of logarithms and exponents are retained. The corresponding derivatives and limits are

$$\frac{d}{dx}\Big\{\log_a(x)\Big\} = \frac{1}{x\log(a)}, \qquad \frac{d}{dx}\Big\{a^x\Big\} = a^x \log(a),$$

$$\lim_{x\to 1} \frac{\log_a(x)}{x-1} = \lim_{t\to 0} \frac{\log_a(1+t)}{t} = \frac{1}{\log(a)} \qquad \text{and} \qquad \lim_{t\to 0} \frac{a^t - 1}{t} = \log(a),$$

in which a does not depend on x or t.