Exercise 1. — Use the limit definition of the derivative to prove the power rule for positive rational exponents.

Solution. — Let p and q be positive integers with no common factor. If $y = x^{\frac{p}{q}}$ and $y' = x'^{\frac{p}{q}}$, then

$$\frac{dy}{dx} = \lim_{x' \to x} \frac{y' - y}{x' - x} = \lim_{x' \to x} \frac{x'^{\frac{p}{q}} - x^{\frac{p}{q}}}{x' - x}.$$

Let $t = x^{\frac{1}{q}}$ and $s = x'^{\frac{1}{q}}$; then $s \to t$ as $x' \to x$ and

$$\begin{aligned}\frac{x'^{\frac{p}{q}} - x^{\frac{p}{q}}}{x' - x} &= \frac{s^p - t^p}{s^q - t^q} \\ &= \frac{(s-t)(s^{p-1} + s^{p-2}t + \cdots + st^{p-2} + t^{p-1})}{(s-t)(s^{q-1} + s^{q-2}t + \cdots + st^{q-2} + t^{q-1})}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{dy}{dx} &= \lim_{s \to t} \frac{s^{p-1} + s^{p-2}t + \cdots + st^{p-2} + t^{p-1}}{s^{q-1} + s^{q-2}t + \cdots + st^{q-2} + t^{q-1}} \\ &= \frac{pt^{p-1}}{qt^{q-1}} = \frac{p}{q} t^{p-q} \\ &= \frac{p}{q} x^{\frac{p-q}{q}}.\end{aligned}$$

Exercise 2. — Use the limit definition of the derivative to prove the power rule for negative rational exponents.

Solution. — Let p and q be positive integers with no common factor. If $y = x^{-\frac{p}{q}}$ and $y' = x'^{-\frac{p}{q}}$, then

$$\frac{dy}{dx} = \lim_{x' \to x} \frac{y' - y}{x' - x} = \lim_{x' \to x} \frac{x'^{-\frac{p}{q}} - x^{-\frac{p}{q}}}{x' - x}.$$

Let $t = x^{\frac{1}{q}}$ and $s = x'^{\frac{1}{q}}$; then $s \to t$ as $x' \to x$ and

$$\begin{aligned}\frac{x'^{-\frac{p}{q}} - x^{-\frac{p}{q}}}{x' - x} &= \frac{s^{-p} - t^{-p}}{s^q - t^q} = \frac{-1}{s^p t^p} \cdot \frac{s^p - t^p}{s^q - t^q} \\ &= \frac{-1}{s^p t^p} \cdot \frac{(s-t)(s^{p-1} + s^{p-2}t + \cdots + st^{p-2} + t^{p-1})}{(s-t)(s^{q-1} + s^{q-2}t + \cdots + st^{q-2} + t^{q-1})}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{dy}{dx} &= \lim_{s \to t} \frac{-1}{s^p t^p} \cdot \frac{s^{p-1} + s^{p-2}t + \cdots + st^{p-2} + t^{p-1}}{s^{q-1} + s^{q-2}t + \cdots + st^{q-2} + t^{q-1}} \\ &= \frac{-1}{t^{2p}} \cdot \frac{pt^{p-1}}{qt^{q-1}} = -\frac{p}{q} t^{-(p+q)} \\ &= -\frac{p}{q} x^{-\frac{p+q}{q}}.\end{aligned}$$

Exercise 3. — Use the limit definition of the derivative to compute the derivative of the sine function.

Solution. — Subtracting the sum and difference identities of the sine function, $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$, gives

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha \sin\beta.$$

If $x' = \alpha + \beta$ and $x = \alpha - \beta$, then $\alpha = \frac{1}{2}(x' + x)$ and $\beta = \frac{1}{2}(x' - x)$. Therefore,

$$\begin{aligned}\frac{d}{dx}\Big\{\sin(x)\Big\} &= \lim_{x' \to x} \frac{\sin(x') - \sin(x)}{x' - x} = \lim_{x' \to x} \left\{\cos\left(\tfrac{1}{2}(x' + x)\right) \cdot \frac{\sin\left(\frac{1}{2}(x' - x)\right)}{\frac{1}{2}(x' - x)}\right\} \\ &= \cos\left(\tfrac{1}{2}(x + x)\right) \cdot 1 = \cos(x).\end{aligned}$$

Exercise 4. — Use the limit definition of the derivative to compute the derivative of the cosine function.

Solution. — Subtracting the sum and difference identities of the cosine function, $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$, gives

$$\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin\alpha \sin\beta.$$

If $x' = \alpha + \beta$ and $x = \alpha - \beta$, then $\alpha = \frac{1}{2}(x' + x)$ and $\beta = \frac{1}{2}(x' - x)$. Therefore,

$$\begin{aligned}\frac{d}{dx}\Big\{\cos(x)\Big\} &= \lim_{x' \to x} \frac{\cos(x') - \cos(x)}{x' - x} = \lim_{x' \to x} \left\{-\sin\left(\tfrac{1}{2}(x' + x)\right) \cdot \frac{\sin\left(\frac{1}{2}(x' - x)\right)}{\frac{1}{2}(x' - x)}\right\} \\ &= -\sin\left(\tfrac{1}{2}(x + x)\right) \cdot 1 = -\sin(x).\end{aligned}$$

Exercise 5. — Use the limit definition of the derivative to compute the derivative of the tangent function.

Solution. — The difference identity of the tangent function is

$$\tan(x' - x) = \frac{\sin(x')\cos(x) - \cos(x')\sin(x)}{\cos(x')\cos(x) + \sin(x')\sin(x)} = \frac{\tan(x') - \tan(x)}{1 + \tan(x')\tan(x)},$$

after dividing and multiplying by $\cos(x')\cos(x)$, or equivalently

$$\tan(x') - \tan(x) = \frac{1 + \tan(x')\tan(x)}{\cos(x' - x)} \cdot \sin(x' - x).$$

Therefore,

$$\begin{aligned}\frac{d}{dx}\Big\{\tan(x)\Big\} &= \lim_{x' \to x} \frac{\tan(x') - \tan(x)}{x' - x} = \lim_{x' \to x} \left\{\frac{1 + \tan(x')\tan(x)}{\cos(x' - x)} \cdot \frac{\sin(x' - x)}{x' - x}\right\} \\ &= \frac{1 + \tan(x)\tan(x)}{\cos(0)} \cdot 1 = 1 + \tan^2(x).\end{aligned}$$