Exercise 1. — State and prove the continuity of differentiable functions.

Solution. — Where y is a differentiable function of x, it is also continuous.

Proof. — Where y is a differentiable function of x,

$$\lim_{x'\to x} y' = \lim_{x'\to x}\left\{\frac{y'-y}{x'-x}\cdot(x'-x)+y\right\} = \frac{dy}{dx}\cdot 0 + y = y,$$

which shows the continuity of y. □

Exercise 2. — State and prove the linearity of the derivative.

Solution. — Where u and v are differentiable functions of x, so is $y = \alpha u + \beta v$, in which α and β are real numbers; moreover,

$$\frac{dy}{dx} = \alpha\frac{du}{dx} + \beta\frac{dv}{dx}.$$

Proof. — By a direct calculation,

$$\begin{aligned}\frac{dy}{dx} &= \lim_{x'\to x}\frac{y'-y}{x'-x} = \lim_{x'\to x}\frac{(\alpha u' + \beta v') - (\alpha u + \beta v)}{x'-x} \\ &= \lim_{x'\to x}\left\{\alpha\cdot\frac{u'-u}{x'-x} + \beta\cdot\frac{v'-v}{x'-x}\right\} = \alpha\frac{du}{dx} + \beta\frac{dv}{dx}.\end{aligned}$$ □

Exercise 3. — State and prove the product rule for differentiation.

Solution. — Where u and v are differentiable functions of x, so is $y = uv$, and

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}.$$

Proof. — The product rule follows from the computation

$$\begin{aligned}\frac{dy}{dx} &= \lim_{x'\to x}\frac{y'-y}{x'-x} = \lim_{x'\to x}\frac{u'v' - uv' + uv' - uv}{x'-x} \\ &= \lim_{x'\to x}\left\{\frac{u'-u}{x'-x}\cdot v' + u\cdot\frac{v'-v}{x'-x}\right\} = \frac{du}{dx}v + u\frac{dv}{dx},\end{aligned}$$

in which $v' \to v$ as $x' \to x$ by the continuity of differentiable functions. □

Exercise 4. — State and prove the reciprocal rule for differentiation.

Solution. — Where u is a differentiable function of x, so is $y = 1/u$ if $u \neq 0$, and

$$\frac{dy}{dx} = -\frac{1}{u^2}\frac{du}{dx}.$$

Proof. — The continuity of differentiable functions implies that $u' \to u$ as $x' \to x$, from which it follows that $u' \neq 0$ if x' is sufficiently near, but not equal to, x. Besides this point, the proof is a direct calculation:

$$\frac{dy}{dx} = \lim_{x'\to x}\frac{y'-y}{x'-x} = \lim_{x'\to x}\left\{\left(\frac{1}{u'} - \frac{1}{u}\right)\cdot\frac{1}{x'-x}\right\} = \lim_{x'\to x}\left\{\frac{-1}{u'u}\cdot\frac{u'-u}{x'-x}\right\} = -\frac{1}{u^2}\frac{du}{dx}.$$ □

Exercise 5. — State and prove the quotient rule for differentiation.

Solution. — Where u and v are differentiable functions, so is $y = u/v$ if $v \neq 0$, and

$$\frac{dy}{dx} = \frac{du}{dx}\cdot\frac{1}{v} - \frac{u}{v^2}\cdot\frac{dv}{dx}.$$

Proof. — The continuity of differentiable functions implies that $v' \to v$ as $x' \to x$, from which it follows that $v' \neq 0$ if x' is sufficiently near, but not equal to, x. Besides this point, the proof is a direct calculation:

$$\begin{aligned}\frac{dy}{dx} &= \lim_{x'\to x}\frac{y'-y}{x'-x} = \lim_{x'\to x}\left\{\left(\frac{u'}{v'} - \frac{u}{v'} + \frac{u}{v'} - \frac{u}{v}\right)\cdot\frac{1}{x'-x}\right\} \\ &= \lim_{x'\to x}\left\{\frac{u'-u}{x'-x}\cdot\frac{1}{v'} - \frac{u}{v'v}\cdot\frac{v'-v}{x'-x}\right\} = \frac{du}{dv}\cdot\frac{1}{v} - \frac{u}{v^2}\cdot\frac{dv}{dx}.\end{aligned}$$ □

Note. — The quotient rule is a consequence of the product and reciprocal rules.

Exercise 6. — State and prove the chain rule for differentiation.

Solution. — Where y is a differentiable function of x and z is a differentiable function of y, z is a differentiable function of x, and

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}.$$

Proof. — Where y is a differentiable function of x,

$$\frac{dy}{dx} = \lim_{x'\to x}\frac{y'-y}{x'-x}, \quad\text{or}\quad y'-y = \left(\frac{dy}{dx} + \varepsilon\right)(x'-x), \quad\text{in which } \varepsilon \to 0 \text{ as } x' \to x.$$

Likewise, where z is a differentiable function of y,

$$\frac{dz}{dy} = \lim_{y'\to y}\frac{z'-z}{y'-y}, \quad\text{or}\quad z'-z = \left(\frac{dz}{dy} + \eta\right)(y'-y), \quad\text{in which } \eta \to 0 \text{ as } y' \to y.$$

Combining these results gives (for $x' \neq x$),

$$z'-z = \left(\frac{dz}{dy} + \eta\right)\left(\frac{dy}{dx} + \varepsilon\right)(x'-x), \qquad\text{or}\qquad \frac{z'-z}{x'-x} = \left(\frac{dz}{dy} + \eta\right)\left(\frac{dy}{dx} + \varepsilon\right).$$

As $x' \to x$, $\varepsilon \to 0$ and $\eta \to 0$ (for $y' \to y$ by the continuity of differentiable functions). Therefore,

$$\frac{dz}{dx} = \lim_{x'\to x}\frac{z'-z}{x'-x} = \lim_{\eta\to 0}\left(\frac{dz}{dy} + \eta\right)\cdot\lim_{\varepsilon\to 0}\left(\frac{dy}{dx} + \varepsilon\right) = \frac{dz}{dy}\frac{dy}{dx}.$$ □