DIFFERENTIATION

§ 1. **Definition.** — The derivative of y with respect to x is denoted by $\frac{dy}{dx}$ and defined by

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x' \to x} \frac{y' - y}{x' - x}. \tag{1}$$

Equivalently,

$$y' - y = \left(\frac{dy}{dx} + \varepsilon \right)(x' - x) \qquad \text{where} \quad \varepsilon \to 0 \quad \text{as} \quad x' \to x. \tag{2}$$

The domain of the derivative $\frac{dy}{dx}$ is the set of all real numbers x for which the limit (1) is defined. On this set y is called a *differentiable* function of x. Where this is so it is plain that $y' \to y$ as $x' \to x$; *i.e.*, a function is differentiable where it is continuous. (Caution: A function may be continuous at every real number and differentiable at no real number, so the converse fails spectacularly.)

§ 2. **Basic properties of the derivative.** — The basic arithmetical properties of the derivative are

$$\frac{d}{dx}(\alpha u + \beta v) = \alpha \frac{du}{dx} + \beta \frac{dv}{dx}, \qquad \frac{d}{dx}(uv) = \frac{du}{dx} v + u \frac{dv}{dx} \qquad \text{and} \qquad \frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\frac{dv}{dx},$$

in which α, β are real numbers, and u, v are differentiable functions of x; in the last equation $v \neq 0$. The basic property of the derivative with respect to composition is

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx},$$

wherever y is a differentiable function of x and z is a differentiable function of y. (Proofs of these properties are in http://jac.tips/nya/dl.pdf.)

§ 3. **Basic differentiation formulæ.** — The most basic differentiation formulæ are:

$$\frac{d}{dx}\left\{x^r\right\} = r x^{r-1}; \qquad \frac{d}{dx}\left\{\log(x)\right\} = \frac{1}{x}; \qquad \frac{d}{dx}\left\{e^x\right\} = e^x;$$

$$\frac{d}{dx}\left\{\sin(x)\right\} = \cos(x); \qquad \frac{d}{dx}\left\{\cos(x)\right\} = -\sin(x); \qquad \frac{d}{dx}\left\{\tan(x)\right\} = 1 + \tan^2(x) = \sec^2(x).$$

(Derivations are in http://jac.tips/nya/df.pdf and http://jac.tips/algebra/logexp.pdf.) Derivatives of other trigonometric functions, and of logarithmic and exponential functions with other (possibly variable) bases, may be computed using the basic properties of the derivative and the following identities (in which $u > 0$ and $u \neq 1$):

$$\log_u(v) = \frac{\log(v)}{\log(u)}; \qquad u^v = e^{v \log(u)}; \qquad \csc(x) = \frac{1}{\sin(x)}; \qquad \sec(x) = \frac{1}{\cos(x)}; \qquad \cot(x) = \frac{1}{\tan(x)}.$$

§ 4. **Disposable formulæ.** — These may be committed to memory, or deduced as indicated above:

$$\frac{d}{dx}\left\{\log_a(x)\right\} = \frac{1}{x \log(a)}; \qquad \frac{d}{dx}\left\{a^x\right\} = a^x \log(a); \qquad \frac{d}{dx}\left\{\csc(x)\right\} = -\csc(x)\cot(x);$$

$$\frac{d}{dx}\left\{\sec(x)\right\} = \sec(x)\tan(x); \qquad \frac{d}{dx}\left\{\cot(x)\right\} = -\csc^2(x).$$

§ 5. **Logarithmic differentiation.** — This refers to uses of the identity $\frac{dy}{dx} = y \frac{d}{dx}\left\{\log|y|\right\}$.

§ 6. **Implicit differentiation.** — Let $\mathscr{C} = \{(x,y) \colon Z = k\}$, where k is a real number and Z depends on x and y. Write Z_x for the derivative of Z with respect to x, and write Z_y for the derivative of Z with respect to y. If Z_x, Z_y are continuous at a point on $\mathscr{C}$ where $Z_y \neq 0$, then $\mathscr{C}$ defines y as a differentiable function of x near (*i.e.*, within a circle containing) this point, where

$$\frac{dy}{dx} = -\frac{Z_x}{Z_y}.$$

This is obtained by writing $0 = Z(x',y') - Z(x,y) = \big(Z_y(x',y) + \varepsilon'\big)(y' - y) + \big(Z_x(x,y) + \varepsilon\big)(x' - x)$ using (2), and then rearranging and computing the limit (1). Beware that there are much more cumbersome ways to compute the derivative, and you should not expect to have the time to waste by using them.