SOME EXERCISES INVOLVING DIFFERENTIAL EQUATIONS

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Exercise 1. — Determine all curves which contain the point (a, b) and whose subtangents all have length c. (A subtangent is the projection onto the x-axis of the part of a tangent from the point of contact to the x-axis.)

Exercise 2. — Find all curves in the first quadrant with the property that the tangent chord bounded by the coordinate axes is always divided by the point of contact in a fixed proportion.

Exercise 3. — Find all smooth curves with the property that the area between the curve and the x-axis on any interval is equal to the length of the curve on that interval.

Exercise 4. — A body was found floating face down in a lake. At $11h$ the body's temperature was $23°$C, and when it was removed from the water at noon its temperature was $21°$C. Assume that the temperature of the lake is a constant $13°$C, and that the body had a normal temperature of $37°$C before going into the water. When was the body dumped?

Exercise 5. — Initially a tank contains 5 kilograms of salt dissolved in 1000 litres of water. Brine containing 7 grams of salt per litre flows into the tank at a constant rate of 7 litres per minute. Another brine, containing 17 grams of salt per litre, flows into the tank at a constant rate of 3 litres per minute. The contents of the tank are kept thoroughly mixed at all times, as the tank is drained at a constant rate of 10 litres per minute. When does the mass of salt in the tank exceed eight kilograms? How much salt is in the tank after five hours?

Exercise 6. — Initially a tank contains 8 kilograms of salt dissolved in 200 litres of water. Brine containing 15 grams of salt per litre flows into the tank at a constant rate of 4 litres per minute. The contents of the tank are kept thoroughly mixed at all times, as the tank is drained at a constant rate of 12 litres per minute. Find an expression for the mass (in kilograms) of salt in the tank after t minutes.

Exercise 7. — Find an equation defining the curve which contains the origin and each of whose normal lines passes through the point $(3, 4)$.

Exercise 8. — During the night of 14 March it began to snow, and the snow fell steadily for many hours. At midnight a crew began clearing snow from a long straight road at a constant rate. They cleared one kilometre during the first two hours, but it took them five more hours to clear the next kilometre. When did it begin snowing?

Exercise 9. — Find all curves with the property that if the normal line is drawn at any point P on the curve (not on the x-axis), the length of projection onto the x-axis of the part of the normal between P and the x-axis is constant.

Exercise 10. — Find all curves with the property that if the normal line is drawn at any point P on the curve (and on neither coordinate axis), then the part of the normal line between P and the x-axis is divided by the y-axis in a fixed proportion.

Exercise 11. — Find all curves with the property that if a line is drawn from the origin to any point (x, y) on the curve and not on the x axis, and then a tangent line is drawn to the curve at that point and extended to meet the x axis, the result is an isosceles triangle with equal sides meeting at (x, y).

Exercise 12. — The growth of a population is jointly proportional to its size and the difference between the carrying capacity of its environment and its size. Initially the population is one-quarter of the carrying capacity, and after one year it is one-third of the carrying capacity. After how long is the population growing most quickly?

Exercise 13. — Find a single differential equation which is satisfied by all circles which are tangent to the x-axis at the origin.

Exercise 14. — Show that if ω is a positive real number and

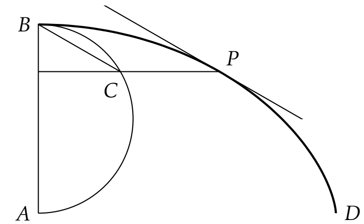
$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

for all real values of t, then there are unique real numbers A and B such that $x = A\cos(\omega t) + B\sin(\omega t)$ for all real values of t. Use this result to find the solution of the initial value problem

$$\frac{d^2x}{dt^2} + 4x = e^{-t}\sin(t); \qquad \left.\frac{dx}{dt}\right|_{t=0} = 3; \qquad x\Big|_{t=0} = -2.$$

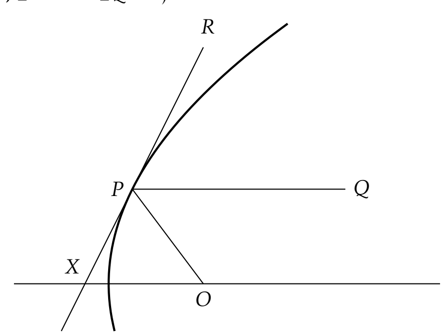
Exercise 15. — A curve rises from the origin into the first quadrant so that the area under the curve between $O(0, 0)$ and $P(x, y)$ is proportional to the area of the rectangle with opposite vertices O and P. Find an equation of the curve.

Exercise 16. — For a semicircle with diameter AB, let $\mathscr{C}$ be the curve which contains B and whose tangent line at any point P is parallel to the chord BC, where C lies on the semicircle and PC is perpendicular to AB. (See the figure.)



Find an equation of $\mathscr{C}$, and the endpoint D of $\mathscr{C}$ where AD is perpendicular to AB. Find the length of $\mathscr{C}$ (from B to D), and the area of the region enclosed by $\mathscr{C}$, the diameter AB and the line segment AD.

Exercise 17. — Find all curves with the property that if a (non-horizontal) line segment is drawn from a fixed point O to any point P on the curve, then the acute angle between the tangent at P and the segment is equal to the acute angle between the tangent at P and the horizontal line through P. (In the figure, $\angle XPO = \angle QPR$.)



Solution to exercise 1. — If $c < 0$, or if $b = 0$ and $c > 0$, then there is no such curve, and if $c = 0$ then the curve is the vertical line defined by $x = a$. It remains to treat the non-degenerate situations, in which $c > 0$ and $b \neq 0$.

Let ξ be the x-intercept of the tangent line to such a curve at the point (x, y), where $y \neq 0$; then the length of the subtangent at (x, y) is $|x - \xi| = c > 0$. Hence,

$$\frac{dy}{dx} = \frac{y}{x - \xi} = \pm\frac{y}{c}, \qquad \text{and therefore} \qquad y = Ae^{\pm x/c},$$

for some real number A (since the differential equation is an equation of exponential growth/decay). As the curve contains the point (a, b), it follows that $b = Ae^{\pm a/c}$, or $A = be^{\mp a/c}$. Therefore, the curve is defined by $y = be^{\pm(x-a)/c}$.

Solution to exercise 2. — Let η be the y-intercept of the tangent line to such a curve at the point $P(x, y)$. By similarity, the (positive) ratio $\rho = y/(\eta - y)$ is constant. Therefore,

$$\frac{dy}{dx} = \frac{y - \eta}{x} = -\frac{y}{\rho x}, \quad \text{and so} \quad \rho x^{\rho-1} y + x^{\rho}\frac{dy}{dx} = 0, \quad \text{or} \quad \frac{d}{dx}(x^{\rho} y) = 0.$$

Hence, there is a positive real number α such that $xy^{\rho} = \alpha$ for $x, y > 0$.

Solution to exercise 3. — Since the reflection in the x-axis of such a curve is again such a curve, it is sufficient to consider curves for which y is a positive function of x. Then (for $a \leqslant x$)

$$\int_a^x y\, d\xi = \int_a^x \sqrt{1 + \left(\frac{dy}{d\xi}\right)^2}\, d\xi, \qquad \text{or} \qquad y = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

by the (first) fundamental theorem of calculus. One such curve is defined by $y = 1$. Otherwise, rearranging and integrating gives

$$1 = \frac{1}{\sqrt{y^2 - 1}}\frac{dy}{dx}, \qquad \text{and hence} \qquad x - \alpha = \log\left(y + \sqrt{y^2 - 1}\right),$$

or $e^{x-\alpha} = y + \sqrt{y^2 - 1}$, for a real number α. Subtracting y and squaring gives

$$e^{2(x-\alpha)} - 2ye^{x-\alpha} = -1, \qquad \text{or} \qquad y = \tfrac{1}{2}(e^{x-\alpha} + e^{\alpha-x}).$$

Nothing serious depends on whether the curve defined by

$$y = \begin{cases} \frac{1}{2}(e^{x-\alpha} + e^{\alpha-x}) & \text{if } x < \alpha, \text{ and} \\ 1 & \text{if } x \geqslant \alpha \end{cases}$$

(or its reflection in the vertical line defined by $x = \alpha$) should count as smooth.

Solution to exercise 4. — If T denotes the temperature (in degrees C) of the body after t hours in the lake and t_0 denotes the number of hours before noon that the body was dumped, then Newton's law of cooling implies that there is a real number a such that

$$\frac{dT}{dt} = a(T - 13), \qquad \text{or equivalently} \qquad \frac{d}{dt}(T - 13) = a(T - 13),$$

which is an exponential equation, so $T - 13 = 24e^{at}$, as $T = 37$ when $t = 0$. Since $T = 23$ when $t = t_0 - 1$ and $T = 21$ when $t = t_0$, it follows that $8 = 24e^{at_0}$ and $10 = 24e^{a(t_0-1)}$, and dividing gives gives $e^a = \frac{4}{5}$. Therefore $\left(\frac{4}{5}\right)^{t_0} = \frac{1}{3}$, and since $\left(\frac{4}{5}\right)^5 = \frac{1024}{3125}$ is just less than $\frac{1}{3}$, it follows that the body was dumped shortly after 7*h*. More precisely (but most likely artificially so), since $60 \cdot \log(1/3)/\log(4/5) \approx 295.40$, the body was dumped roughly four hours, fifty-five minutes and twenty-four seconds before noon, or approximately 36 seconds after 7*h*04.

Solution to exercise 5. — If m denotes the mass (in kilograms) of salt in the tank after t minutes then

$$\frac{d}{dt}(m - 10) = \frac{7}{1000}\cdot 7 + \frac{17}{1000}\cdot 3 - \frac{m}{1000}\cdot 10 = -\tfrac{1}{100}(m - 10),$$

so $10 - m = 5e^{-t/100}$, since the differential equation is an exponential equation and $m = 5$ when $t = 0$. So the mass of salt in the tank exceeds eight kilograms when $5e^{-t/100} < 2$, or $t > 100\log(5/2)$. (This is a little more than ninety-one and a half minutes.) After three hours, there are $10 - 5e^{-3}$ kilograms of salt in the tank. (This is approximately $9\frac{3}{4}$ kilograms.)

Solution to exercise 6. — If m denotes the mass (in kilograms) of salt in the tank after t minutes then

$$\frac{dm}{dt} = \frac{15}{1000}\cdot 4 - \frac{m}{200 - 8t}\cdot 12 = \frac{3}{50} - \frac{3m}{2(25 - t)}.$$

On dividing by $(25 - t)^{3/2}$ and rearranging, the product rule for differentiation gives

$$\frac{d}{dt}\left\{\frac{m}{(25 - t)^{3/2}}\right\} = \frac{3}{50(25 - t)^{3/2}},$$

and so

$$\frac{m}{(25 - t)^{3/2}} = \frac{3}{25(25 - t)^{1/2}} + C, \qquad \text{or} \qquad m = \tfrac{3}{25}(25 - t) + C(25 - t)^{3/2}.$$

Since $m = 8$ when $t = 0$, it follows that $8 = 3 + 125C$, or $C = \frac{1}{25}$. Therefore, there are

$$m = \tfrac{1}{25}(25 - t)(3 + \sqrt{25 - t})$$

kilograms of salt in the tank after t minutes (where $0 \leqslant t \leqslant 25$).

Solution to exercise 7. — If $P(x, y)$ is any point on the curve for which $y \neq 4$, then

$$\frac{dy}{dx} = \frac{3 - x}{y - 4}, \qquad \text{or} \qquad x - 3 + (y - 4)\frac{dy}{dx} = 0,$$

so the curve is defined by $(x - 3)^2 + (y - 4)^2 = 25$. In other words, the curve is a circle with centre $(3, 4)$ and radius 5.

Solution to exercise 8. — Let x denote the number of kilometres cleared by the crew, t the number of hours after midnight that the crew has been working and t_0 the number of hours before midnight that it began to snow. The depth of the snow on the uncleared part of the road is proportional to $t + t_0$; it is also inversely proportional to $\frac{dx}{dt}$, so there is a positive real number a such that

$$a(t + t_0)\frac{dx}{dt} = 1, \qquad \text{or equivalently} \qquad \frac{d}{dx}(t + t_0) = a(t + t_0).$$

Since this is an exponential differential equation for which $x = 0$ when $t = 0$, it follows that $t = t_0(\alpha^x - 1)$, where $\alpha = e^a$. Now $x = 1$ when $t = 2$ and $x = 2$ when $t = 7$, so $2 = t_0(\alpha - 1)$ and $7 = t_0(\alpha^2 - 1)$, and dividing gives $\frac{7}{2} = \alpha + 1$, or $\alpha = \frac{5}{2}$. Again using the fact that $x = 1$ when $t = 2$ gives $2 = t_0\left(\frac{5}{2} - 1\right) = \frac{3}{2}t_0$, or $t_0 = \frac{4}{3}$. Therefore, it began snowing at (about) 22*h*40 on the night of March 14.

Solution to exercise 9. — Let λ be the length of the projection as described. If $\lambda = 0$ then the curve is a straight line parallel to the x-axis. Otherwise, let ξ be the x-intercept of the normal line to the curve at the point $P(x, y)$, where $y \neq 0$. Then

$$\frac{dy}{dx} = \frac{\xi - x}{y} = \pm\frac{\lambda}{y}, \qquad \text{or} \qquad y\frac{dy}{dx} = \pm\lambda, \qquad \text{and thus} \qquad y^2 = \mu \pm 2\lambda x,$$

for some real number μ. So every such curve is either a straight line parallel to the x-axis, or else a parabola with vertical directrix and focus on the x-axis.

Solution to exercise 10. — Let ξ denote the x-intercept of the normal line to such a curve at the point $P(x, y)$, where $x \neq 0$ and $y \neq 0$. By similarity, the (positive) ratio $\alpha - 1 = -\xi/x$ is independent of P. Hence,

$$\frac{dy}{dx} = \frac{\xi - x}{y} = -\frac{\alpha x}{y}, \qquad \text{or} \qquad \alpha x\frac{dy}{dx} + y = 0, \qquad \text{and so} \qquad \alpha x^2 + y^2 = \beta,$$

for some positive real number β. Therefore, every such curve is an ellipse centred at the origin (whose major axis is vertical, since $\alpha > 1$).

Solution to exercise 11. — Let ξ be the x-intercept of the tangent to such a curve at the point $P(x,y)$. Since the triangle formed by the origin, P and $(\xi,0)$ is isosceles, with sides of equal length meeting at P, it follows that $\xi = 2x$, and thus

$$\frac{dy}{dx} = \frac{y}{x-\xi} = -\frac{y}{x}, \qquad \text{and so} \qquad x\frac{dy}{dx} + y = 0, \qquad \text{or} \qquad \frac{d}{dx}(xy) = 0.$$

Hence, there is a (non-zero) real number α such that $xy = \alpha$; so every such curve is a rectangular hyperbola whose asymptotes are the coordinate axes.

Solution to exercise 12. — If p denotes the population and m the (constant) carrying capacity, then there is a positive real number $\alpha = km$ such that

$$\frac{dp}{dt} = kp(m-p), \qquad \text{or equivalently} \qquad \left(\frac{1}{p} + \frac{1}{m-p}\right)\frac{dp}{dt} = \alpha,$$

which implies that

$$\log\frac{m-p}{p} = -\alpha t + C, \qquad \text{or} \qquad \frac{m}{p} - 1 = Ae^{-\alpha t}.$$

If t is measured in years, then the values of the population initially and after one year give $3 = A$ and $2 = 3e^{-\alpha}$, so $\frac{m}{p} - 1 = 3\left(\frac{2}{3}\right)^t$. The population is increasing most rapidly when $\frac{dp}{dt} = kp(m-p)$ assumes its largest value, *i.e.*, when p is one-half of m. This occurs when $\left(\frac{2}{3}\right)^t = \frac{1}{3}$, or after $\log(1/3)/\log(2/3)$ years. (Since $365\big(\log(1/3)/\log(2/3) - 2\big) \approx 258.97$, this occurs by the two hundred and fifty-ninth day of the third year.)

Solution to exercise 13. — Such a circle is defined by $x^2 + (y-a)^2 = a^2$, or $x^2 + y^2 = 2ay$, for some real number a. Therefore,

$$2x = (2a - 2y)\frac{dy}{dx}, \quad \textit{i.e.}, \quad 2xy = (x^2 - y^2)\frac{dy}{dx}, \quad \text{or} \quad \frac{dy}{dx} = \frac{2xy}{x^2 - y^2}.$$

Solution to exercise 14. — Let A and ωB be, respectively, the values of x and $\frac{dx}{dt}$ corresponding to $t = 0$, and let $z = x - A\cos(\omega t) - B\sin(\omega t)$. Plainly $z = 0$ and $\frac{dz}{dt} = 0$ if $t = 0$, and

$$\frac{d^2z}{dt^2} + \omega^2 z = 0$$

for all real values of t. Since

$$\frac{d}{dt}\left\{\left(\frac{dz}{dt}\right)^2 + (\omega z)^2\right\} = 2\frac{dz}{dt}\left\{\frac{d^2z}{dt^2} + \omega^2 z\right\} = 0,$$

it follows that

$$\left(\frac{dz}{dt}\right)^2 + (\omega z)^2 = 0$$

for all real values of t, since it is constant and equal to zero if $t = 0$. Therefore $z = 0$, or equivalently, $x = A\cos(\omega t) + B\sin(\omega t)$, for all real values of t, as required.

To solve the initial value problem, observe that if $y = \frac{1}{10}e^{-t}(\cos(t) + 2\sin(t))$, then

$$\frac{dy}{dt} = \tfrac{1}{10}e^{-t}(\cos(t) - 3\sin(t)), \qquad \text{and} \qquad \frac{d^2y}{dt^2} = \tfrac{1}{5}e^{-t}(\sin(t) - 2\cos(t)),$$

so that

$$\frac{d^2y}{dt^2} + 4y = e^{-t}\sin(t).$$

By the first part of the solution to this exercise, it follows that the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 4x = e^{-t}\sin(t)$$

is given by $x = \frac{1}{10}e^{-t}(\cos(t) + 2\sin(t)) + A\cos(2t) + B\sin(2t)$. Since

$$x\bigg|_{t=0} = A + \tfrac{1}{10} \qquad \text{and} \qquad \frac{dx}{dt}\bigg|_{t=0} = 2B + \tfrac{1}{10},$$

the initial condition requires that $-2 = A + \frac{1}{10}$ and $3 = 2B + \frac{1}{10}$, or $A = -\frac{21}{10}$ and $B = \frac{29}{20}$.

Solution to exercise 15. — There is a positive real number ϖ such that

$$(1+\varpi)\int_0^x y\,d\xi = xy, \quad \textit{i.e.}, \quad (1+\varpi)y = y + x\frac{dy}{dx}, \quad \text{or} \quad \frac{d}{dx}(x^{-\varpi}y) = 0.$$

Therefore, there is a positive real number α such that $x^{-\varpi}y = \alpha$, or $y = \alpha x^{\varpi}$.

Solution to exercise 16. — Choose coordinates in which the semicircle has equation $x = \sqrt{a^2 - y^2}$, where $a > 0$. If $-a < y < a$ and $P(x,y)$ lies on $\mathscr{C}$, then

$$\frac{dx}{dy} = \frac{\sqrt{a^2 - y^2}}{y - a} = -\frac{a+y}{\sqrt{a^2 - y^2}},$$

as the tangent at P is parallel to the chord joining $B(0,a)$ and $C(\sqrt{a^2-y^2}, y)$. Therefore, the curve $\mathscr{C}$ is defined by

$$x = -\int_a^y \frac{a+y}{\sqrt{a^2 - y^2}}\,dy = a\arccos(y/a) + \sqrt{a^2 - y^2}.$$

Although the integral is improper its convergence is standard (and resolved by integration, which gives the endpoints of $\mathscr{C}$ as well). From this it is plain that the point D on $\mathscr{C}$ at which the segment AD is perpendicular to the diameter AB has y coordinate $-a$ and x coordinate πa. The differential equation of $\mathscr{C}$ implies that

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{a+y}{a-y} = \frac{2a}{a-y},$$

so the length of $\mathscr{C}$ is equal to

$$\sqrt{2a}\int_{-a}^{a}\frac{dy}{\sqrt{a-y}} = \lim_{\eta\to a^-} -2\sqrt{2a(a-y)}\bigg|_{-a}^{\eta} = 4a.$$

The differential equation of $\mathscr{C}$ implies that the area of the region between $\mathscr{C}$ and the line which is perpendicular to AB and contains B is equal to

$$\int_{x=0}^{x=\pi a}(a-y)\,dx = \int_{y=-a}^{y=a}\sqrt{a^2-y^2}\,dy = \tfrac{1}{2}\pi a^2,$$

since the latter integral is the area of the semicircle with diameter AB. The area of the rectangle formed by AB and AD is equal to $2\pi a^2$, so the area of the region enclosed by AB, AD and $\mathscr{C}$ is equal to $\frac{3}{2}\pi a^2$.

Solution to exercise 17. — Choose coordinates so that O is the origin and the x-intercepts of the tangents are to the left of O. If ξ is the x-intercept of the tangent line to such a curve at $P(x,y)$, where $y > 0$, then

$$\frac{dx}{dy} = \frac{x-\xi}{y}, \qquad \text{or} \qquad \xi = x - y\frac{dx}{dy}.$$

Let $x = yz$, so that

$$\frac{dx}{dy} = z + y\frac{dz}{dy}, \qquad \text{and hence} \qquad \xi = -y^2\frac{dz}{dy}.$$

From $\angle XPO = \angle QPR = \angle OXP$, it follows that $|OX| = |OP|$, and therefore

$$\xi^2 = x^2 + y^2, \quad \textit{i.e.}, \quad y^4\left(\frac{dz}{dy}\right)^2 = y^2z^2 + y^2, \quad \text{or} \quad \frac{1}{\sqrt{z^2+1}}\frac{dz}{dy} = \frac{1}{y}.$$

Thus, there is a positive real number p such that $2p = e^c$ and

$$\log\left(z + \sqrt{z^2+1}\right) + c = \log y, \qquad \text{or} \qquad 2pz + 2p\sqrt{z^2+1} = y.$$

Subtracting $2pz$ and squaring gives

$$4p^2 = y^2 - 4pyz = y^2 - 4px, \qquad \text{or} \qquad y^2 = 4p(x+p).$$

Hence, every such curve is a parabola with vertical directrix and focus at O.