Infinite series (summary)

1. **Infinite series.** To a(n *infinite) series* is associated its sequence $\{s_n\}_{n\geqslant 1}$ of partial sums

$$\sum_{n=k}^{\infty} a_n = \underbrace{a_k + a_{k+1} + a_{k+2} + \cdots + a_{k+n-1}}_{s_n} + \cdots \qquad (\dagger)$$

If $\lim s_n$ is defined then $s = \lim s_n$ is *the sum of* (†), and (†) *converges to* s, or *is equal to* s (in the orthodox sense). If $\lim s_n$ is undefined then (†) *diverges* (to ∞, to $-\infty$, or oscillates, as appropriate). The linearity of limits implies that an infinite series converges if, and only if, any non-zero multiple of it does, and that the sum of a convergent series and a divergent series is a divergent series. The convergence of a series is not affected by the introduction or suppression of a finite number of terms; sub/superscripts on the sign of summation may be left implicit when considering convergence (not the sum or partial sums). A sequence $\{x_n\}$ *eventually* has a property if there is a positive integer N such that $\{x_n\}_{n\geqslant N}$ has that property.

2. **Vanishing condition.** $\sum a_n$ diverges unless $\{a_n\}$ vanishes (*i.e.*, if $\sum a_n$ converges then $\lim a_n = 0$).

3. **Telescoping series.** If $a_n = b_n - b_{n+1}$ for $n \geqslant k$, then $s_n = b_k - b_{k+n}$ for $n \geqslant k$, and therefore $\sum_{n=k}^{\infty} a_n = b_k - b$ if $\lim b_n = b$, and $\sum a_n$ diverges if $\lim b_n$ is undefined. If $b_n = ar^n/(1-r)$, this is a

4. **Geometric series.** If $r \neq 1$ then $a + ar + ar^2 + \cdots + ar^{n-1} = a(1-r^n)/(1-r)$, and therefore

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if} \quad |r| < 1, \quad \text{and} \quad \sum_{n=0}^{\infty} ar^n \quad \text{diverges if} \quad |r| \geqslant 1 \text{ and } a \neq 0.$$

5. **Fundamental criterion for series of non-negative terms.** If $a_n \geqslant 0$ for $n \geqslant N$, then $\sum a_n$ converges if, and only if, its sequence of partial sums is bounded.

Example. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges since $1/(2^k+1) + \cdots + 1/(2^{k+1}) > 1/2$ if $k \geqslant 1$, and therefore $1 + \frac{1}{2} + \cdots + \frac{1}{2^n} \geqslant 1 + \frac{1}{2}n$ if $n \geqslant 0$.

Example. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges since $1/(2^k)^2 + \cdots + 1/(2^{k+1}-1)^2 < 1/2^k$ if $k \geqslant 1$, and therefore $0 < s_n \leqslant s_{2^n-1} < 2(1-1/2^n) < 2$ if $n \geqslant 1$, which implies that $\{s_n\}$ is bounded.

6. **Scales of convergence.** Reflecting on the method used to investigate the preceding examples reveals that if $\{a_n\}$ is eventually (non-negative and) monotonic, then

$$\sum a_n \quad \text{converges} \qquad \text{if, and only if,} \qquad \sum 2^n a_{2^n} \quad \text{converges.} \qquad \text{(condensation)}$$

Condensing a p-series gives a geometric series, with $r = 2^{1-p}$, and condensing a logarithmic p-series gives a non-zero multiple of a p-series with the same value of p. This yields the following scales of convergence.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad \text{converges if } p > 1, \quad \text{and diverges if } p \leqslant 1. \qquad (p\text{-series})$$

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \qquad \text{converges if } p > 1, \quad \text{and diverges if } p \leqslant 1. \qquad (\text{logarithmic } p\text{-series})$$

7. **Comparison Test.** Suppose that $0 \leqslant a_n \leqslant b_n$ if $n \geqslant N$.

   a. If $\sum b_n$ converges then $\sum a_n$ converges. b. If $\sum a_n$ diverges then $\sum b_n$ diverges.

8. **Limit Comparison Test.** Suppose that $a_n \geqslant 0$ and $b_n > 0$ if $n \geqslant N$, and that $\lim \dfrac{a_n}{b_n} = \lambda$.

   a. If λ is a non-negative real number and $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
   b. If λ is a positive real number or ∞ and $\sum b_n$ is divergent, then $\sum a_n$ is divergent.

In particular, if λ is a positive real number then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

9. **Integral Test.** Suppose that $f(x) \geqslant f(y) > 0$ if $a \leqslant x < y$, and that $a_n = f(n)$ if $n \geqslant k$. Then

$$\sum_{n=k}^{\infty} a_n \qquad \text{converges if, and only if,} \qquad \int_a^{\infty} f(x)\,dx \qquad \text{converges.}$$

Moreover, if $\sum_{n=k}^{\infty} a_n = s$ and $a \leqslant k+n-1$, then

$$s_n + \int_{k+n}^{\infty} f(x)\,dx < s < s_n + \int_{k+n-1}^{\infty} f(x)\,dx.$$

10. **Ratio and Root Tests.** Suppose that $a_n > 0$ ($a_n \geqslant 0$ in the Root Test) if $n \geqslant N$, and that

$$\lim \frac{a_{n+1}}{a_n} = \varrho \quad \text{(Ratio Test)} \qquad \text{or} \qquad \lim \sqrt[n]{a_n} = \varrho \quad \text{(Root Test)}.$$

   a. If $\varrho < 1$ then $\sum a_n$ converges. b. If $\varrho > 1$ then $\lim a_n = \infty$, and so $\sum a_n$ diverges.

If $\varrho = 1$ then neither test is conclusive (if $a_n = 1/n^p$ then in each case $\varrho = 1$).

11. **Series of arbitrary terms.** $\sum a_n$ is *absolutely convergent* (*converges absolutely*) if $\sum |a_n|$ converges. Since $a_n = |a_n| - \big(|a_n| - a_n\big)$ and $0 \leqslant |a_n| - a_n \leqslant 2|a_n|$, the Comparison Test implies that $\sum a_n$ converges if it converges absolutely. $\sum a_n$ is *conditionally convergent* (*converges conditionally*) if $\sum a_n$ converges but $\sum |a_n|$ diverges. Many examples of conditionally convergent series are provided by the

12. **Alternating Series Test.** If $a_n \geqslant a_{n+1} > 0$ for $n \geqslant N$ and $\lim a_n = 0$ then $\sum (-1)^n a_n$ converges. (In place of $(-1)^n$ can be any sequence whose partial sums are bounded; so *e.g.*, $\sum (\sin n)/n$ converges.) Moreover, the absolute difference between the sum of the series and any partial sum which includes a_N is at most the absolute value of the next term of the series. Together with the scales of convergence, the Alternating Series Test implies that:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \begin{cases} \text{converges absolutely if } p > 1, \\ \text{converges conditionally if } 0 < p \leqslant 1, \text{ and} \\ \text{diverges if } p \leqslant 0; \end{cases} \qquad (\text{alternating } p\text{-series})$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^p} \begin{cases} \text{converges absolutely if } p > 1, \text{ and} \\ \text{converges conditionally if } p \leqslant 1. \end{cases} \qquad (\text{alternating logarithmic } p\text{-series})$$

If the preceding items are inconclusive, the following may help.

13. **Raabe/Duhamel Test.** Suppose that $a_n > 0$ if $n \geqslant N$, and that

$$\lim \left\{ n\left(\frac{a_{n+1}}{a_n} - 1\right)\right\} = \varpi.$$

   a. If $\varpi < -1$ then $\sum a_n$ converges. b. If $\varpi > 0$ then $\sum (-1)^n a_n$ diverges.
   c. If $-1 < \varpi < 0$ then $\sum (-1)^n a_n$ converges and $\sum a_n$ diverges.

If $\varpi = -1$ then $\sum(-1)^n a_n$ converges but $\sum a_n$ may converge or diverge (*e.g.*, if $a_n = 1/\big(n(\log n)^p\big)$, then $\varpi = -1$). If $\varpi = 0$ then $\sum(-1)^n a_n$ may converge or diverge (*e.g.*, if $a_n = (\log n)^{\pm 1}$, then $\varpi = 0$). This is the second (the Ratio Test is the first) in an infinite list of tests based on a comparison principle for ratios; the effect of each test is to place a scale in the gap of ignorance left by the preceding test.

14. **Gauß Test.** Suppose that if $n \geqslant N$, then $a_n > 0$ and

$$\frac{a_{n+1}}{a_n} = \frac{n^\ell + An^{\ell-1} + B_2 n^{\ell-2} + \cdots + B_\ell}{n^\ell + an^{\ell-1} + b_2 n^{\ell-2} + \cdots + b_\ell}, \qquad \text{where } \ell \text{ is a positive integer.}$$

   a. If $A - a < -1$ then $\sum a_n$ converges. b. If $A - a \geqslant 0$ then $\sum (-1)^n a_n$ diverges.
   c. If $-1 \leqslant A - a < 0$ then $\sum (-1)^n a_n$ converges and $\sum a_n$ diverges.

The Gauß Test has no exceptions because of the special form of a_{n+1}/a_n. It is useful for determining the convergence of a wide class of power series at the endpoints of their intervals of convergence.