Determinants

Linear Algebra I

201-NYC-05

The idea of determinant (linear maps)

The determinant of a linear map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is the oriented factor by which T scales n-dimensional volumes.

The concept immediately yields many determinants.

Examples

- The determinant of any rotation is 1.
- The determinant of any shear is 1.
- The determinant of any projection onto a proper subspace is 0.
- The determinant of any reflection in a plane in $\mathbb{R}^3$ is -1.
- The determinant of any reflection in a line in $\mathbb{R}^3$ is 1.
- The determinant of an expansion along one axis by a factor a is a.
- The determinant of dilation by a factor a is a^n.
- The determinant of any singular linear map $\mathbb{R}^n \to \mathbb{R}^n$ is 0.

The idea of determinant (matrices)

The $1 \times 1 \times \cdots \times 1$ cube formed by $\mathbf{e}_1, \mathbf{e}_2, , \ldots, \mathbf{e}_n$, whose oriented n-dimensional volume is plainly 1, is mapped by $\mathbf{x} \rightsquigarrow A\mathbf{x} \colon \mathbb{R}^n \to \mathbb{R}^n$ onto

$$\wp(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \{\, t_1\mathbf{a}_1 + \cdots + t_n\mathbf{a}_n \colon 0 \leqslant t_j \leqslant 1, \text{ for } j = 1, 2, \ldots, n \,\},$$

which is called an n-*dimensional parallelotope*, or n-*parallelotope*.

The determinant of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ in $\mathbb{R}^n$ is the oriented n-dimensional volume of the parallelotope $\wp(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ formed by $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

For $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^n$, the determinant is written as

$$\det\begin{pmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n\end{pmatrix}, \qquad \begin{vmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n\end{vmatrix}, \qquad \det(A) \qquad \text{or} \qquad |A|,$$

where $A = \begin{pmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n\end{pmatrix}$ is the $n \times n$ matrix whose columns are $\mathbf{a}_1, \ldots, \mathbf{a}_n$. The notation with vertical lines is frequently used in numerical calculations and *does not indicate an absolute value*.

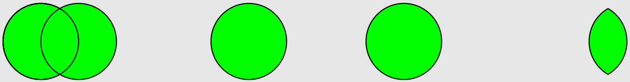
The idea of determinant (volume)

What is n-*dimensional volume*? For small values of n, it is familiar.

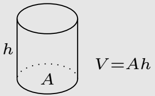
- 1-dimensional volume is *length*.
- 2-dimensional volume is *area*.
- 3-dimensional volume is *volume* in the usual sense.

For larger values of n it is unfamiliar, but with familiar properties. The n-dimensional volume of a figure $\mathscr{F}$ will be denoted by $\mathrm{Vol}_n(\mathscr{F})$.

- If $\mathscr{F}$ is contained in an $(n-1)$-flat then $\mathrm{Vol}_n(\mathscr{F}) = 0$.
- For any vector $\mathbf{u}$ in $\mathbb{R}^n$, $\mathrm{Vol}_n(\mathbf{u} + \mathscr{F}) = \mathrm{Vol}_n(\mathscr{F})$.
- $\mathrm{Vol}_n(\mathscr{F}_1 \cup \mathscr{F}_2) = \mathrm{Vol}_n(\mathscr{F}_1) + \mathrm{Vol}_n(\mathscr{F}_2) - \mathrm{Vol}_n(\mathscr{F}_1 \cap \mathscr{F}_2)$.



- If $\mathbf{n}$ is orthogonal to an $(n-1)$-flat containing $\mathscr{F}$, then $\mathrm{Vol}_n(\{\, t\mathbf{n} + \mathscr{F} : 0 \leqslant t \leqslant 1 \,\}) = \|\mathbf{n}\| \, \mathrm{Vol}_{n-1}(\mathscr{F})$.



The idea of determinant (orientation)

What does *oriented* mean? For small values of n, it is familiar.

- In the number line (*i.e.*, in $\mathbb{R}$), orientation distinguishes right (the positive direction) from left (the negative direction).
- In the plane (*i.e.*, in $\mathbb{R}^2$), orientation distinguishes counterclockwise (positive) rotations from clockwise (negative) rotations.
- In space (*i.e.*, in $\mathbb{R}^3$), orientation distinguishes right handed (positive) triples of vectors from left handed (negative) triples of vectors.

For larger values of n it is unfamiliar, but with familiar properties.

- There are two orientations (called *positive* and *negative*; the choice of which orientation counts as positive is conventional).
- Multiplying a vector by a positive scalar preserves orientation.
- Multiplying a vector by a negative scalar reverses orientation.
- Interchanging two vectors reverses orientation.

The idea of determinant (examples)

Determinants of order one ($\mathbb{R}$)

Determinants of order one are almost too simple to discuss.

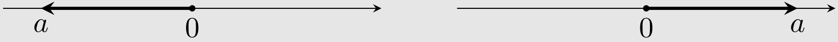
If a is a real number, the parallelotope

$$\wp(a) = \{\, ta\colon 0 \leqslant t \leqslant 1 \,\},$$

is the directed segment from 0 to a, and its oriented length is a:

$$\det(a) = a.$$

The orientation of a determinant of order one indicates a positive or negative direction along the number line.



$(a < 0)$ $(a > 0)$

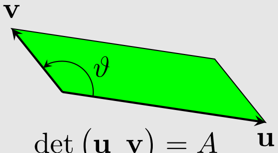
The idea of determinant (examples)

Determinants of order two ($\mathbb{R}^2$)

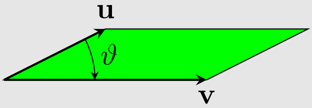
For linearly independent vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$, the parallelotope

$$\wp\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix} = \{\, s\mathbf{u} + t\mathbf{v} \colon 0 \leqslant s \leqslant 1,\, 0 \leqslant t \leqslant 1 \,\},$$

is the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$. If A is the area of $\wp\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix}$ and ϑ is the shortest angle from $\mathbf{u}$ to $\mathbf{v}$, then $\det\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix} = A$ if ϑ is positive and $\det\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix} = -A$ if ϑ is negative.



$\det\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix} = A$



$\det\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix} = -A$

The orientation of a determinant of order two indicates a counterclockwise (positive) or clockwise (negative) shortest angle from the first vector to the second vector.

The idea of determinant (examples)

Determinants of order three ($\mathbb{R}^3$)

For linearly independent vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, the parallelotope

$$\wp\left(\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}\right) = \{\, r\mathbf{u} + s\mathbf{v} + t\mathbf{w} \colon 0 \leqslant r \leqslant 1,\, 0 \leqslant s \leqslant 1,\, 0 \leqslant t \leqslant 1 \,\}.$$

is the *parallelepiped* (3-dimensional box) formed by $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

If V denotes the volume of $\wp\left(\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}\right)$, then

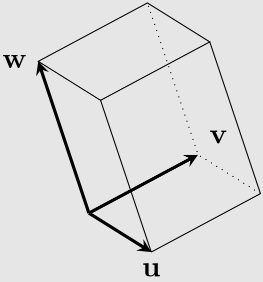
$$\det\left(\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}\right) = \begin{cases} V & \text{if } \{\mathbf{u},\mathbf{v},\mathbf{w}\} \text{ is right handed, and} \\ -V & \text{if } \{\mathbf{u},\mathbf{v},\mathbf{w}\} \text{ is left handed.} \end{cases}$$

This is illustrated on the next page.

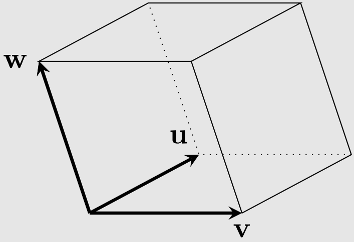
The idea of determinant (examples)

Determinants of order three ($\mathbb{R}^3$), illustrations

In each figure, V is the volume of the parallelepiped formed by the linearly independent vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$.



$$\det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix} = V$$



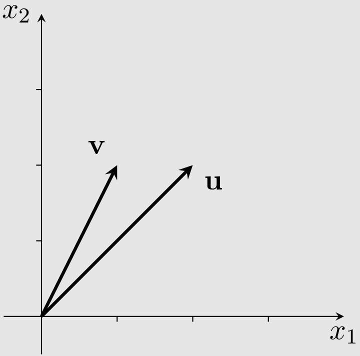
$$\det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix} = -V$$

The orientation of a determinant of order three indicates a right handed or left handed triple of vectors.

The idea of determinant (example)

A sample calculation in $\mathbb{R}^2$

Here is a calculation of $\det\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix} = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix}$ using the idea of determinant.



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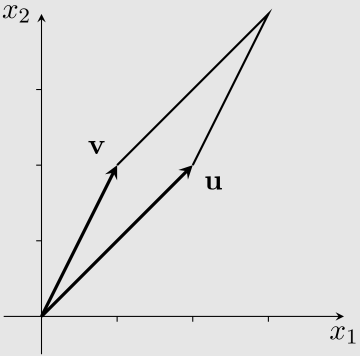

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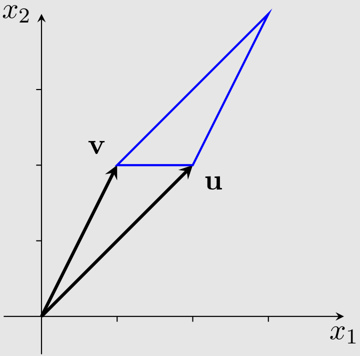

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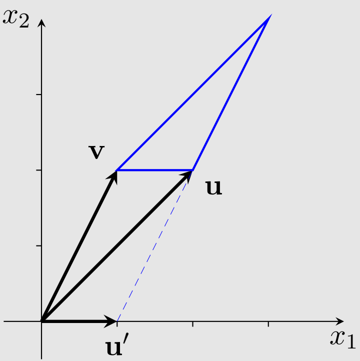

$$\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} \qquad [\,\mathbf{c}_1 \leftarrow \mathbf{c}_1 - \mathbf{c}_2\,]$$

The idea of determinant (example)

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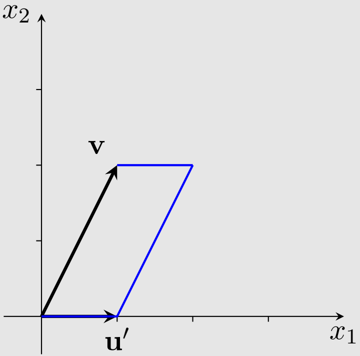

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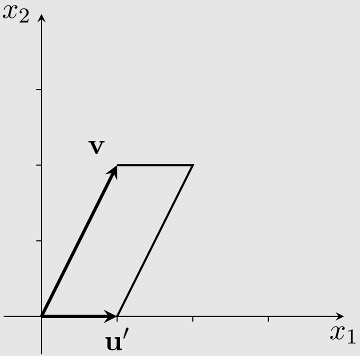

$$\begin{aligned} &\begin{vmatrix}2 & 1\\ 2 & 2\end{vmatrix} \qquad [\,\mathbf{c}_1 \leftarrow \mathbf{c}_1 - \mathbf{c}_2\,] \\ =&\begin{vmatrix}1 & 1\\ 0 & 2\end{vmatrix}\end{aligned}$$

The idea of determinant (example)

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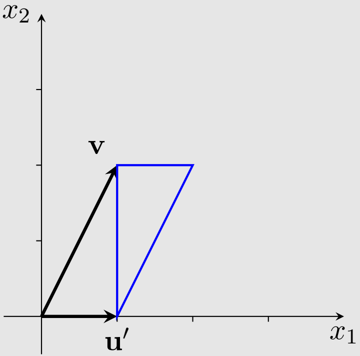

$$\begin{vmatrix}2 & 1\\ 2 & 2\end{vmatrix} \qquad [\,\mathbf{c}_1 \leftarrow \mathbf{c}_1 - \mathbf{c}_2\,]$$

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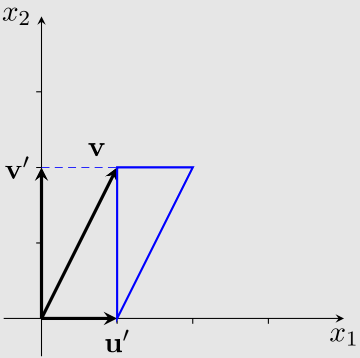

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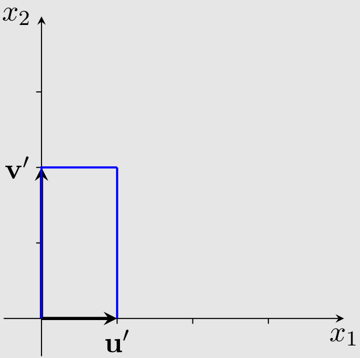

$$
\begin{aligned}
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= &\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} && [\,\mathbf{c}_2 \leftarrow \mathbf{c}_2 - \mathbf{c}_1\,]
\end{aligned}
$$

The idea of determinant (example)

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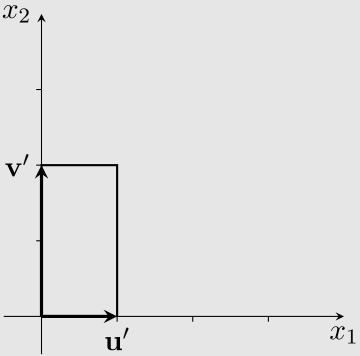

$$\begin{aligned}
&\begin{vmatrix}2 & 1\\ 2 & 2\end{vmatrix} & &[\,\mathbf{c}_1 \leftarrow \mathbf{c}_1 - \mathbf{c}_2\,]\\
=&\begin{vmatrix}1 & 1\\ 0 & 2\end{vmatrix} & &[\,\mathbf{c}_2 \leftarrow \mathbf{c}_2 - \mathbf{c}_1\,]\\
=&\begin{vmatrix}1 & 0\\ 0 & 2\end{vmatrix}
\end{aligned}$$

The idea of determinant (example)

A sample calculation in $\mathbb{R}^2$

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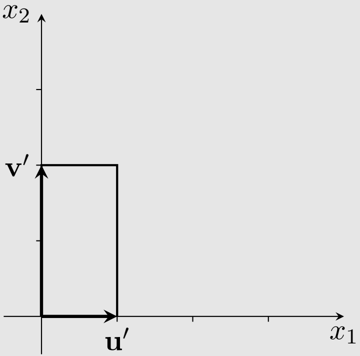


$$
\begin{aligned}
&\begin{vmatrix}2 & 1\\ 2 & 2\end{vmatrix} && [\,\mathbf{c}_1 \leftarrow \mathbf{c}_1 - \mathbf{c}_2\,]\\
&= \begin{vmatrix}1 & 1\\ 0 & 2\end{vmatrix} && [\,\mathbf{c}_2 \leftarrow \mathbf{c}_2 - \mathbf{c}_1\,]\\
&= \begin{vmatrix}1 & 0\\ 0 & 2\end{vmatrix}\\
&= 1 \cdot 2\\
&= 2
\end{aligned}
$$

Characteristic properties of determinants (notation)

The basic idea implies a few properties of determinants which characterize them uniquely. The properties require a bit of notation.

If A is an $m \times n$ matrix, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ and $1 \leqslant i, j \leqslant n$, then $A_j(\mathbf{x})$ is the matrix obtained by replacing column j of A by $\mathbf{x}$ and $A_{i,j}(\mathbf{x}, \mathbf{y})$ is the matrix obtained by replacing column i of A by $\mathbf{x}$ and column j of A by $\mathbf{y}$.

Example

If $A = \begin{pmatrix} 1 & 0 & 2 & 4 \\ 3 & 3 & 4 & 3 \\ 2 & 1 & 7 & 5 \\ 3 & 2 & 1 & 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$ then

$$A_2(\mathbf{x}) = \begin{pmatrix} 1 & x_1 & 2 & 4 \\ 3 & x_2 & 4 & 3 \\ 2 & x_3 & 7 & 5 \\ 3 & x_3 & 1 & 2 \end{pmatrix} \quad \text{and} \quad A_{1,3}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} x_1 & 0 & y_1 & 4 \\ x_2 & 3 & y_2 & 3 \\ x_3 & 1 & y_3 & 5 \\ x_4 & 2 & y_4 & 2 \end{pmatrix}$$

Characteristic properties of determinants (alternating)

There are two properties of the determinant of order n,

$$\det(A) = \det\begin{pmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n\end{pmatrix},$$

which imply all of its algebraic properties, and a third property which, together with the first two, determines its value uniquely. The properties arise from reflection on the basic idea. First, the basic idea implies that the determinant of a singular linear map $\mathbb{R}^n \to \mathbb{R}^n$ must be 0; thus:

Determinants are alternating

If two columns of A are equal then $\det(A) = 0$; equivalently,

$$\det A_i(\mathbf{a}_j) = 0, \qquad \text{or} \qquad \det A_{i,j}(\mathbf{x}, \mathbf{x}) = 0,$$

whenever $1 \leqslant i, j \leqslant n$, $i \neq j$, and $\mathbf{x}$ is any vector in $\mathbb{R}^n$.

A function of k vectors in $\mathbb{R}^n$ with this property is called *alternating*.

Characteristic properties of determinants (n-linearity)

The determinant of order n is n-linear

For each j, $1 \leqslant j \leqslant n$, the mapping $\mathbb{R}^n \to \mathbb{R}$ defined by $\mathbf{x} \rightsquigarrow \det A_j(\mathbf{x})$ is linear; that is

$$\det A_j(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha \det A_j(\mathbf{x}) + \beta \det A_j(\mathbf{y}),$$

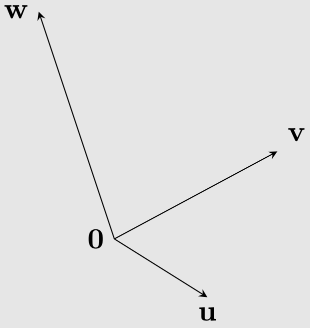
for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{R}$, or equivalently

$$\begin{aligned} \det A_j(\mathbf{x} + \mathbf{y}) &= \det A_j(\mathbf{x}) + \det A_j(\mathbf{y}), \text{ and} && [\text{additivity}] \\ \det A_j(\alpha\mathbf{x}) &= \alpha \det A_j(\mathbf{x}). && [\text{preservation of scaling}] \end{aligned}$$

Notice that n-linearity implies that $\det(\alpha A) = \alpha^n \det(A)$ for any scalar α.

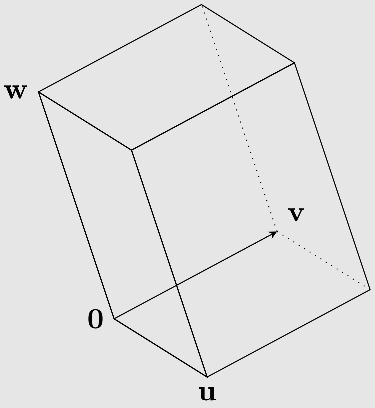
A function of k vectors in $\mathbb{R}^n$ with this property is called *k-linear*.

Preservation of scaling (illustration)



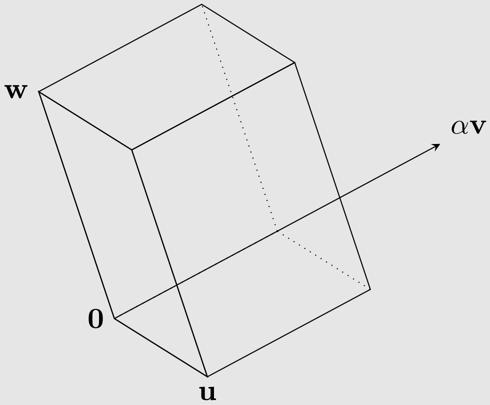
$$\begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix}$$

Preservation of scaling (illustration)



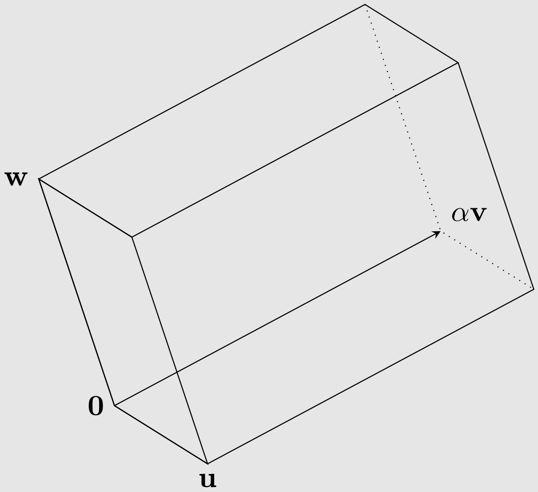
$$\det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix}$$

Preservation of scaling (illustration)



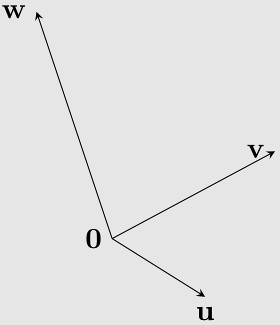
$$\begin{pmatrix}\mathbf{u} & \alpha\mathbf{v} & \mathbf{w}\end{pmatrix} \qquad \det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix}$$

Preservation of scaling (illustration)



$$\det\begin{pmatrix}\mathbf{u} & \alpha\mathbf{v} & \mathbf{w}\end{pmatrix} = \alpha\det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix}$$

Additivity (illustration)



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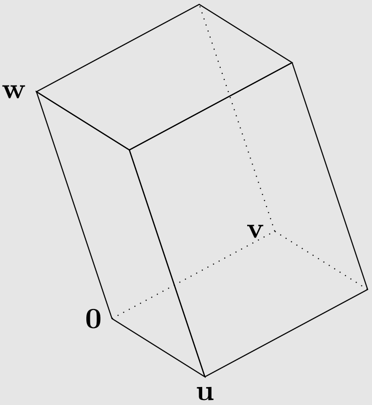

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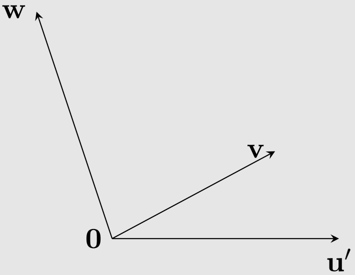
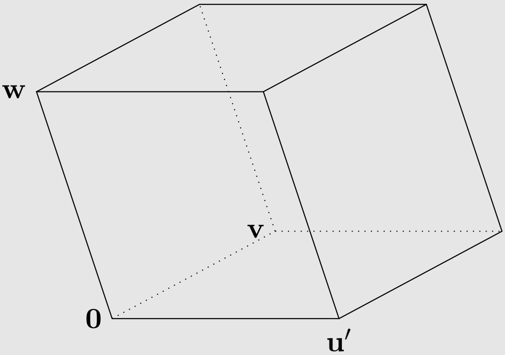
Additivity (illustration)



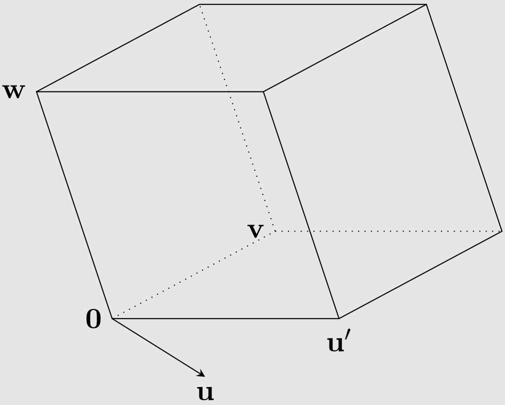
$$\begin{pmatrix} \mathbf{u}' & \mathbf{v} & \mathbf{w} \end{pmatrix}$$

Additivity (illustration)



$$\det\begin{pmatrix}\mathbf{u}' & \mathbf{v} & \mathbf{w}\end{pmatrix}$$

Additivity (illustration)



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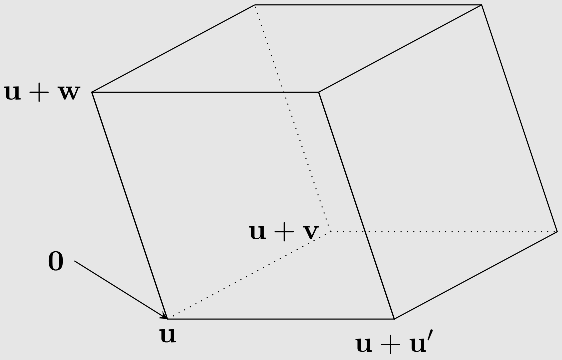
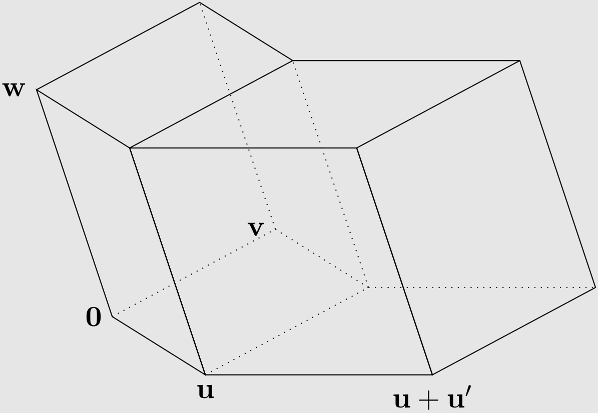


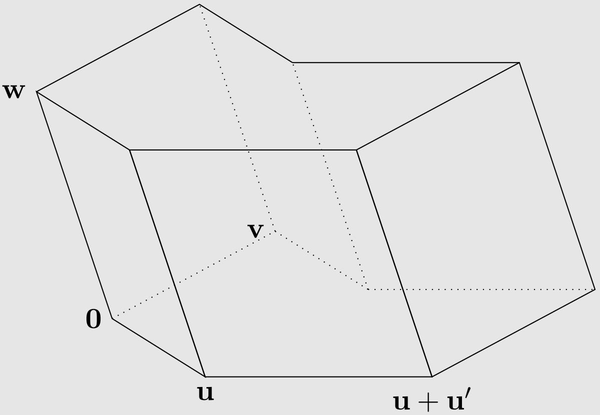
$$\det\begin{pmatrix}\mathbf{u}' & \mathbf{v} & \mathbf{w}\end{pmatrix}$$

Additivity (illustration)



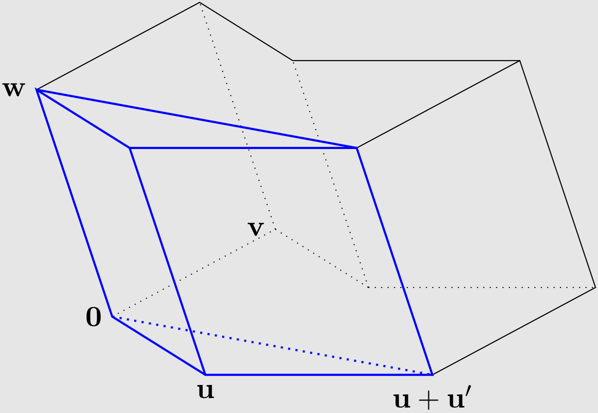
$$\det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix} + \det\begin{pmatrix}\mathbf{u}' & \mathbf{v} & \mathbf{w}\end{pmatrix}$$

Additivity (illustration)



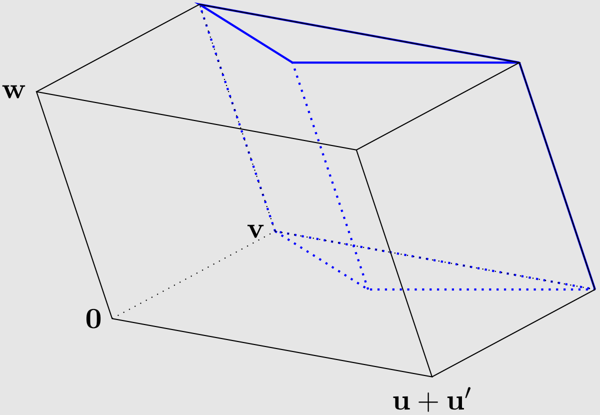
$$\det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix} + \det\begin{pmatrix}\mathbf{u}' & \mathbf{v} & \mathbf{w}\end{pmatrix}$$

Additivity (illustration)



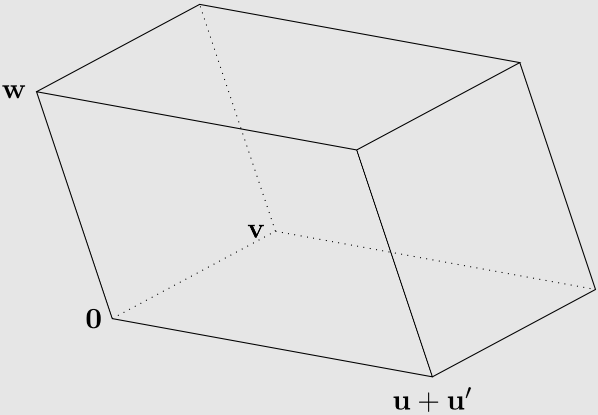
$$\det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix} + \det\begin{pmatrix}\mathbf{u}' & \mathbf{v} & \mathbf{w}\end{pmatrix}$$

Additivity (illustration)



$$\det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix} + \det\begin{pmatrix}\mathbf{u}' & \mathbf{v} & \mathbf{w}\end{pmatrix}$$

Additivity (illustration)



$$\det\begin{pmatrix}\mathbf{u}+\mathbf{u}' & \mathbf{v} & \mathbf{w}\end{pmatrix} = \det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix} + \det\begin{pmatrix}\mathbf{u}' & \mathbf{v} & \mathbf{w}\end{pmatrix}$$

Characteristic properties of determinants

The two characteristic properties of the determinant of order n give rise to its orientation as follows. If $\mathbf{x}$, $\mathbf{y}$ are vectors in $\mathbb{R}^n$ and $i \neq j$, then

$$\begin{aligned} 0 &= \det A_{i,j}(\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) \\ &= \det A_{i,j}(\mathbf{x}, \mathbf{x}) + \det A_{i,j}(\mathbf{x}, \mathbf{y}) + \det A_{i,j}(\mathbf{y}, \mathbf{x}) + \det A_{i,j}(\mathbf{y}, \mathbf{y}) \\ &= \det A_{i,j}(\mathbf{x}, \mathbf{y}) + \det A_{i,j}(\mathbf{y}, \mathbf{x}), \end{aligned}$$

where the first and third equations use the alternating property and the second equation uses n-linearity. Therefore,

$$\det A_{i,j}(\mathbf{x}, \mathbf{y}) = -\det A_{i,j}(\mathbf{y}, \mathbf{x}).$$

In words: If two columns of A are interchanged then the determinant is multiplied by -1.

Notice that the displayed equation is a property of any alternating k-linear function of k vectors in $\mathbb{R}^n$.

Characterization of the determinant

Main Theorem

For every $n \geqslant 1$, and every scalar δ, there is a unique alternating n-linear map $\Delta\colon \mathrm{M}_{n\times n} \to \mathbb{R}$ such that $\Delta(I_n) = \delta$.

The parallelotope in $\mathbb{R}^n$ formed by $\mathbf{e}_1, \dots, \mathbf{e}_n$ is a $1 \times \cdots \times 1$ (hyper)cube, so the determinant of I_n should be 1. This yields the following

Definition

The determinant of order n is the unique alternating n-linear map $\det\colon \mathrm{M}_{n\times n} \to \mathbb{R}$ such that $\det(I_n) = 1$.

The proof of the theorem applies to $n \times k$ matrices: the dimension of the space of alternating k-linear maps on $\mathbb{R}^n$ is $\binom{n}{k} = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdots \frac{n+1-k}{k}$, the number of ways of choosing k vectors from n vectors. For example, the determinant of a 4×2 matrix is a vector in a linear space of dimension $\frac{4}{1} \cdot \frac{3}{2} = 6$. These objects are important, and we'll study one example later.

Characterization of the determinant (proof)

Proof of the theorem

Recall that $\mathbf{a}_j = a_{1j}\mathbf{e}_1 + \cdots a_{nj}\mathbf{e}_n$, where a_{ij} is the entry in row i and column j of A. A map $\Delta\colon \mathrm{M}_{n\times n} \to \mathbb{R}$ is alternating, n-linear, and maps $I_n = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{pmatrix}$ to δ if, and only if,

$$\begin{aligned}
\Delta(A) &= \sum_{\sigma} a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}\, \Delta\begin{pmatrix} \mathbf{e}_{\sigma(1)} & \mathbf{e}_{\sigma(2)} & \cdots & \mathbf{e}_{\sigma(n)} \end{pmatrix} \\
&= \sum_{\sigma} \operatorname{sign}(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}\, \underbrace{\Delta\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{pmatrix}}_{\delta} \\
&= \delta \sum_{\sigma} \operatorname{sign}(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n},
\end{aligned}$$

where the sum is over all permutations $\sigma(1), \sigma(2), \ldots, \sigma(n)$ of $1, 2, \ldots, n$, and $\operatorname{sign}(\sigma)$ is 1 if the number of pairs (i, j) such that $i < j$ and $\sigma(i) > \sigma(j)$ is even, and is -1 otherwise. This proves the theorem.

Properties of determinants (invariance under transposition)

The main theorem implies several properties of determinants which are useful both for calculations and for theoretical investigations.

$\det(A^T) = \det(A)$

Every permutation σ has a unique inverse σ^{-1}, which has the same sign. So $\sum_\sigma \operatorname{sign}(\sigma) \cdots = \sum_{\sigma^{-1}} \operatorname{sign}(\sigma^{-1}) \cdots$; the explicit formula (applied to A, and then to A^T noting the inversion of columns and rows) then gives

$$\begin{aligned}
\det(A) &= \sum_\sigma \operatorname{sign}(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n} \\
&= \sum_{\sigma^{-1}} \operatorname{sign}(\sigma^{-1}) a_{1,\sigma^{-1}(1)} a_{2,\sigma^{-1}(2)} \cdots a_{n,\sigma^{-1}(n)} \\
&= \det(A^T).
\end{aligned}$$

Thus, the determinant is *invariant under transposition*.

Properties of determinants (product preservation)

$\det(AX) = (\det A)(\det X)$

For any $n \times n$ matrix A, the maps $\Delta_1, \Delta_2 \colon \mathrm{M}_{n\times n} \to \mathbb{R}$ defined by

$$\Delta_1(X) = \det(AX) \qquad \text{and} \qquad \Delta_2(X) = (\det A)(\det X)$$

are alternating and n-linear. Since

$$\Delta_1(I_n) = \det(AI_n) = \det(A)$$

and

$$\Delta_2(I_n) = (\det A)(\det I_n) = \det(A),$$

the main theorem implies that $\Delta_1 = \Delta_2$.

Thus, the determinant *preserves products* (or is *multiplicative*).

Properties of determinants

Determinants and elementary column/row operations

I. Since $\det A_j(\mathbf{a}_j + \alpha\mathbf{a}_i) = \det A_j(\mathbf{a}_j) + \alpha \det A_j(\mathbf{a}_i) = \det(A)$, adding a multiple of one column to another column (replacement) does not change a determinant. The determinant is invariant under transposition, so adding a multiple of a row is added to another row does not change a determinant.

II The orientation implies that interchanging two columns multiplies a determinant by -1. The determinant is invariant under transposition, so interchanging two rows multiplies a determinant by -1.

III Multilinearity implies that scaling a column scales a determinant by the same factor. The determinant is invariant under transposition, so scaling a row scales a determinant by the same factor.

The properties for row operations require invariance under transposition, and are not valid for determinants of $n \times k$ matrices if $1 \leqslant k < n$.

Properties of determinants (invertible matrices)

If $\alpha_1\mathbf{a}_1 + \cdots + \alpha_n\mathbf{a}_n = \mathbf{0}$, then scaling column j by α_j, and then adding $\alpha_i\mathbf{a}_i$ to column j for $i \neq j$, gives

$$\alpha_j \det(A) = \det A_j(\alpha_1\mathbf{a}_1 + \cdots + \alpha_n\mathbf{a}_n) = \det A_j(\mathbf{0}) = 0,$$

since the last determinant is obtained by multiplying column j of A by 0.
If A has linearly dependent columns then some α_j is not equal to zero, which implies that $\det(A) = 0$.
On the other hand, if A is invertible then $\det\big(A^{-1}\big)\det(A) = \det(I_n) = 1$, since the determinant preserves products, so $\det(A) \neq 0$, and

$$\det(A^{-1}) = \frac{1}{\det A}.$$

This gives another characterization of invertible matrices.

A is invertible (non-singular) if, and only if, $\det A \neq 0$.

Computing determinants

Recall that a square triangular matrix A is invertible if, and only if, its diagonal entries are non-zero, and replacement operations will bring it to a diagonal form, whose column j is a_{jj} times column j of I_n. Therefore,

the determinant of a triangular matrix is the product of its diagonal entries.

A determinant may be calculated by using column/row operations to obtain a triangular form, keeping track of the effects, and then multiplying the diagonal entries. More generally, if A and D are square, then

$$\det \begin{pmatrix} A & C \\ 0 & D \end{pmatrix} = \det \begin{pmatrix} A & 0 \\ B & D \end{pmatrix} = (\det A)(\det D).$$

since using column/row to bring A into triangular form need not affect D, and likewise using column/row operations to bring D into triangular form need not affect A.

Computing determinants (continued)

Below A is an $i \times j$ matrix (i and j could be zero, with A and C absent) and M is an $n \times n$ matrix, so nj adjacent column interchanges move M to the left, and then ni adjacent row interchanges move M to the top.

$$\begin{vmatrix} A & 0 & C \\ E & M & F \\ B & 0 & D \end{vmatrix} = (-1)^{nj} \begin{vmatrix} 0 & A & C \\ M & E & F \\ 0 & B & D \end{vmatrix} = (-1)^{ni+nj} \begin{vmatrix} M & E & F \\ 0 & A & C \\ 0 & B & D \end{vmatrix}$$

The same calculation would also yield a block triangular determinant if M had only zeros to its left and right; so the result of the previous page gives

$$\begin{vmatrix} A & 0 & C \\ E & M & F \\ B & 0 & D \end{vmatrix} = \begin{vmatrix} A & G & C \\ 0 & M & 0 \\ B & H & D \end{vmatrix} = (-1)^{n(i+j)} \cdot \left|M\right| \cdot \begin{vmatrix} A & C \\ B & D \end{vmatrix}$$

The *sign adjustment* $(-1)^{n(i+j)}$ *for the isolated block* M is -1 if both n and $i+j$ are odd, and is otherwise 1.

Computing determinants (example)

In the calculation below, the isolated blocks and their sign adjustments are coloured cyan, and the zeros isolating the blocks are coloured fuschia.

$$\begin{vmatrix} 5 & 0 & 0 & 8 & -2 & 4 & 2 & 3 & 7 \\ 2 & 4 & 9 & 2 & 2 & 2 & 5 & -4 & -5 \\ 0 & 0 & 0 & 2 & 1 & 3 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 5 & 3 & 4 & 0 \\ 0 & 0 & 0 & 2 & -3 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 9 & 0 \\ 2 & 1 & 3 & 3 & 1 & 1 & 2 & 3 & 2 \\ 4 & 0 & 0 & 4 & 3 & 1 & 5 & -5 & 5 \end{vmatrix} = -1 \cdot \begin{vmatrix} 2 & 1 & 3 & 2 & 2 \\ 1 & 4 & 5 & 3 & 4 \\ 2 & -3 & 0 & 0 & -1 \\ 3 & 0 & 0 & 0 & 7 \\ 3 & 0 & 0 & 0 & 9 \end{vmatrix} \cdot \begin{vmatrix} 5 & 0 & 0 & 7 \\ 2 & 4 & 9 & -5 \\ 2 & 1 & 3 & 2 \\ 4 & 0 & 0 & 5 \end{vmatrix}$$

$$= -1 \cdot \begin{vmatrix} 3 & 2 \\ 5 & 3 \end{vmatrix} \cdot \begin{vmatrix} 2 & -3 & -1 \\ 3 & 0 & 7 \\ 3 & 0 & 9 \end{vmatrix} \cdot 1 \cdot \begin{vmatrix} 4 & 9 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 5 & 7 \\ 4 & 5 \end{vmatrix}$$

$$= -(9 - 10) \cdot -1 \cdot (-3) \cdot \begin{vmatrix} 3 & 7 \\ 3 & 9 \end{vmatrix} \cdot (12 - 9) \cdot (25 - 28)$$

$$= 3 \cdot (27 - 21) \cdot 3 \cdot (-3)$$

$$= -162.$$