## THE ROW REDUCTION ALGORITHM

The row reduction algorithm transforms a matrix into another matrix in which the linear relations satisfied by the columns are transparent, without changing the linear relations between the columns. In this way, it yields all linear relations satisfied by the columns of a matrix, as well as linearly independent columns which generate all linear combinations of the columns a matrix. The algorithm has a *forward phase* and a *backward phase*. A *pivot column* of a matrix is a column which is not a linear combination of the columns to its left. The forward phase identifies the pivot columns of the matrix, and the backward phase expresses the remaining columns as linear combinations of the pivot columns.

**§1. Elementary row operations.** — The basic computations in the row reduction algorithm are called *elementary row operations*, of which there are three types.

I. (Replacement) Replace a row by the sum of that row and a multiple of another row.
II. (Interchange) Interchange two rows.
III. (Scaling) Multiply a row by a non-zero number.

Each type of elementary row operation preserves linear relations between columns, and the effect of an elementary row operation can be reversed by an elementary row operation of the same type. Matrices $A$ and $B$ are called *row equivalent* (notation: $A \sim B$) if one can be obtained from the other by applying a sequence of elementary row operations.

**§2. Forward phase.** — The forward phase has two actions, which are applied recursively.

– If the top entry of the leftmost nonzero column is zero, interchange two rows so as to put a non-zero entry in that position.
– Create zeros in all positions below the top of the leftmost non-zero column by adding multiples of the top row to the rows beneath it, as necessary.

These actions are then applied to the part of the matrix below and to the right of the aforementioned top entry, then to the part below and to the right of next top entry, *&c*.

The result is a matrix with the following properties:

– all zero rows are below all non-zero rows;
– the leading entry in a row is to the right of the leading entry in any row above it.

A matrix with these properties is called an *echelon matrix*, or a matrix in *echelon form*.

**§3. Backward phase.** — The backward phase has two actions, which are performed beginning with the rightmost leading entry, and moving to the left.

– Create zeros above the leading entry by adding multiples of its row to rows above it.
– If a leading entry is not 1, then scale its row so that it becomes 1.

This results in an echelon matrix with the following two additional properties:

– Each leading entry is the only non-zero entry in its column.
– Each leading entry is 1.

Such a matrix is called a *reduced echelon matrix*, or a matrix in *reduced echelon form*.

**§4. Results.** — It is not difficult to prove that ***every matrix is row equivalent to one, and only one, reduced echelon matrix***, and that ***matrices of the same shape are row equivalent if, and only if, their columns satisfy the same linear relations***. The set of all linear combinations of a matrix is called its *column space*. The pivot columns of a matrix are linearly independent and generate its column space. Any linear relation satisfied by the columns of a matrix $A$ can be written in the form $A\mathbf{x} = \mathbf{0}$ ($\mathbf{x}$ is the vector of coefficients of the columns in the relation), and the set of all such vectors $\mathbf{x}$ is called the *null space* of the matrix $A$. The vectors obtained from the relations expressing non-pivot columns as linear combinations of pivot columns, obtained from the reduced echelon form of $A$, are linearly independent and generate the null space of $A$ (using them to eliminate the non-pivot columns from an arbitrary linear relation must yield a trivial relation, since the later involves only pivot columns).

**§5. Example.** — The elementary row operations applied to a matrix are written to its right.

$$A = \begin{pmatrix} 1 & -2 & 0 & 2 & 1 & 0 \\ 4 & -8 & 2 & 6 & 0 & 6 \\ 1 & -2 & 1 & 1 & -1 & 3 \\ -1 & 2 & 3 & -5 & 2 & 0 \\ 2 & -4 & -3 & 7 & 2 & -3 \end{pmatrix} \qquad \begin{aligned} R_2 &\leftarrow R_2 - 4R_1 \\ R_3 &\leftarrow R_3 - R_1 \\ R_4 &\leftarrow R_4 + R_1 \\ R_5 &\leftarrow R_5 - 2R_1 \end{aligned}$$

$$\sim \begin{pmatrix} 1 & -2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & -2 & -4 & 6 \\ 0 & 0 & 1 & -1 & 2 & 3 \\ 0 & 0 & 3 & -3 & 3 & 0 \\ 0 & 0 & -3 & 3 & 0 & -3 \end{pmatrix} \qquad \begin{aligned} R_3 &\leftarrow R_3 - \tfrac{1}{2}R_2 \\ R_4 &\leftarrow R_4 - \tfrac{3}{2}R_2 \\ R_5 &\leftarrow R_5 + \tfrac{3}{2}R_2 \end{aligned}$$

$$\sim \begin{pmatrix} 1 & -2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & -2 & -4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & -9 \\ 0 & 0 & 0 & 0 & -6 & 6 \end{pmatrix} \qquad \begin{aligned} R_3 &\leftrightarrow R_5 \\ R_4 &\leftarrow R_4 + \tfrac{3}{2}R_3 \end{aligned}$$

$$\sim \begin{pmatrix} 1 & -2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & -2 & -4 & 6 \\ 0 & 0 & 0 & 0 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} R_1 &\leftarrow R_1 + \tfrac{1}{6}R_3 \\ R_2 &\leftarrow R_2 - \tfrac{2}{3}R_2 \\ R_3 &\leftarrow -\tfrac{1}{6}R_3 \end{aligned}$$

[The forward phase is complete; the backward phase begins here.]

$$\sim \begin{pmatrix} 1 & -2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad R_2 \leftarrow \tfrac{1}{2}R_2$$

$$\sim \begin{pmatrix} 1 & -2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R$$

[The backward phase is complete; this is the reduced echelon form of the given matrix.]

Write $A = \begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6\end{pmatrix}$ and $R = \begin{pmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 & \mathbf{r}_6\end{pmatrix}$; $\mathbf{r}_1, \mathbf{r}_3, \mathbf{r}_5$ are linearly independent, $\mathbf{r}_2 = -2\mathbf{r}_1$, $\mathbf{r}_4 = 2\mathbf{r}_1 - \mathbf{r}_3$ and $\mathbf{r}_6 = \mathbf{r}_1 + \mathbf{r}_3 - \mathbf{r}_5$. The columns of $A$ and the columns of $R$ satisfy the same linear relations, so $\mathbf{a}_1$, $\mathbf{a}_3$, $\mathbf{a}_5$ are linearly independent, $\mathbf{a}_2 = -2\mathbf{a}_1$, $\mathbf{a}_4 = 2\mathbf{a}_1 - \mathbf{a}_2$ and $\mathbf{a}_6 = \mathbf{a}_1 + \mathbf{a}_3 - \mathbf{a}_5$; moreover, every linear relation satisfied by the columns of $A$ is a linear combination of these last three relations (as explained in §4). The pivot columns of $A$, *i.e.*,

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 4 \\ 1 \\ -1 \\ 2 \end{pmatrix}, \qquad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \\ -3 \end{pmatrix}, \qquad \mathbf{a}_5 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \\ 2 \end{pmatrix}$$

are linearly independent and generate the column space of $A$. The relations can be written as

$$2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}, \qquad -2\mathbf{a}_1 + \mathbf{a}_3 + \mathbf{a}_4 = \mathbf{0} \qquad \text{and} \qquad -\mathbf{a}_1 - \mathbf{a}_3 + \mathbf{a}_5 + \mathbf{a}_6 = \mathbf{0},$$

so the following vectors are linearly independent and generate the null space of $A$:

$$\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$